1. Suppose (x_n) is a bounded sequence and $\lim y_n = 0$. Use the definition of limit to prove $\lim x_n y_n = 0$.

Since (x_n) is bounded, $\exists M > 0 \quad \forall n \quad |x_n| \leq M$

Given $\varepsilon > 0$, since $y_n \to 0$

$\exists k \quad \forall n \geq k \quad |y_n| < \frac{\varepsilon}{M}$ > 0 :)

For $n \geq k$ $\quad |x_n y_n| = |x_n||y_n| < M \frac{\varepsilon}{M} = \varepsilon$:)

2. Suppose (x_n) is sequence in $\mathbf{R}$ that is not bounded above. Prove that (x_n) has a subsequence convergent to $+\infty$.

If (x_n) is not bounded above, then any tail $(x_k, x_{k+1}, \cdots)$ is not bounded above

Pf If not, $\exists k$ $\exists M>0$ $\forall n \geq k$ $x_n \leq M$

Then $\forall n$ $x_n \leq \max\{x_1, x_2, \ldots x_{k-1}, M\}$ ☹

If z_n is increasing and not bounded above, then $z_n \to +\infty$.

Pf Given $\varepsilon>0$, Since $\frac{1}{\varepsilon}$ is not an upper bound for (z_n), $\exists k$ $z_k > \frac{1}{\varepsilon}$

Since (z_n) is increasing, for $n \geq k$ $z_n \geq z_k > \frac{1}{\varepsilon}$ ☺

Since 1 is not an upper bound for (x_n)

$\exists n_1$ $x_{n_1} > 1$

Since (x_n) is not bounded above, the tail

$x_{n_1+1}, x_{n_1+2}, x_{n_1+3}$ is not bounded above,

So $\exists n_2 > n_1$ s.t. $x_{n_2} > \max\{x_{n_1}, 2\}$

Then (x_{n_j}) is an increasing subsequence, not bounded above (each $x_{n_j} > j$)

$\therefore (x_{n_j}) \to +\infty$ ☺

3. Suppose (x_n) is a bounded sequence in $\mathbf{R}$ and $\limsup x_n = \liminf x_n$. What can you conclude? Prove your assertion.

Let $S_k = (x_k, x_{k+1}, \cdots)$ (k-th tail)

$$\forall n \geq k \quad \inf S_k \leq x_n \leq \sup S_k$$

$$\liminf x_n = \lim x_n = \limsup x_n$$

(by squeeze law) ☺

4. Prove that every Cauchy sequence in $\mathbf{R}$ is bounded.

By def of Cauchy seq. with $\varepsilon = 1$

$\exists k \quad \forall n, m \geq k \qquad |x_n - x_m| < 1$

In particular, since $k \geq k$ $\quad \forall m \geq k \quad |x_k - x_m| < 1$

$\therefore \qquad |x_m| - |x_k| \leq |x_k - x_m| < 1$

so $\qquad \forall m \geq k \quad |x_m| \leq |x_k| + 1$

so $\quad \forall m \qquad |x_m| \leq \max\{|x_1|, |x_2|, \dots |x_{k-1}|, |x_k| + 1\}$

:)

5. Suppose $\sum x_n$ is convergent and $\sum y_n$ is divergent. Prove $\sum(x_n + y_n)$ is divergent.

Suppose $\sum(x_n+y_n)$ converges.

then $\displaystyle\lim_{k\to\infty} \sum_{n=1}^{k} (x_n+y_n)$ exists.

Since $\sum x_n$ converges, $\displaystyle\lim_{k\to\infty} \sum_{n=1}^{k} x_n$ exists

$$\sum_{n=1}^{k} y_n = \sum_{n=1}^{k} (x_n+y_n-x_n) = \sum_{n=1}^{k}(x_n+y_n) - \sum_{n=1}^{k} x_n$$

reshuffling finite sums is OK. ◡̈

$$\lim_{k\to\infty} \sum_{n=1}^{k} y_n = \lim_{k\to\infty}\left(\sum_{n=1}^{k}(x_n+y_n) - \sum_{n=1}^{k} x_n\right)$$

$$= \underbrace{\lim_{k\to\infty} \sum_{n=1}^{k}(x_n+y_n)} - \underbrace{\lim_{k\to\infty}\sum_{n=1}^{k} x_n} \text{ exist!}$$

(by the usual algebraic results for limits)

$\therefore$ $\sum y_n$ conv. ◠̈