Midterm 1 / 2023.10.5 / MAT 3013.002 / Foundations of Mathematics

1. If p and q are propositions, the contrapositive tautology is that the proposition $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$. Use a truth table to prove this.

```
f(p,q):=[p,q,not p or q,not q, not p,not(not q)or not p];
s:[f(p,q)]$
range:[true,false]$
for p in range
do for q in range
do s:append(s,[f(p,q)])$
apply(matrix,s);
```

(wxmaxima)

$$f(p,q) := [p, q, \underbrace{\neg p \lor q}_{p \Rightarrow q}, \neg q, \neg p, \underbrace{\neg\neg q \lor \neg p}_{\sim q \Rightarrow \sim p}]$$

p	q	$\neg p \lor q$	$\neg q$	$\neg p$	$q \lor \neg p$
true	true	true	false	false	true
true	false	false	true	false	false
false	true	true	false	true	true
false	false	true	true	true	true

2. If A and B are sets, prove that $A \cup B = A$ if and only if $B \subseteq A$

$\Leftarrow$ Suppose $B \subseteq A$

$\supseteq$ $A \subseteq A \cup B$ (if $x \in A$, $x \in A \cup B$)

$\subseteq$ Let $x \in A \cup B$. If $x \in A$, done.

If $x \in B$, then since $B \subseteq A$, $x \in A$ ☺

$\Rightarrow$ Suppose $A \cup B = A$. Let $x \in B$. Then $x \in A \cup B$, so $x \in A$ ☺

3. Construct an explicit counterexample using finite sets to the (false) proposition that for any sets A and B we have $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$

Let $A = \{1\}$, $B = \{2\}$, so $A \cup B = \{1, 2\}$.

Then $A \cup B \subseteq A \cup B$, so $A \cup B \in \wp(A \cup B)$, but

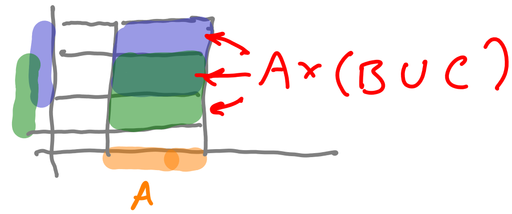
$A \cup B \not\subseteq A$ so $A \cup B \notin \wp(A)$ and $A \cup B \not\subseteq B$, so $A \cup B \notin \wp(B)$,

so $A \cup B \notin \wp(A) \cup \wp(B)$ ☺

Note: Suppose $A \subseteq B$. Then $A \cup B = B$.

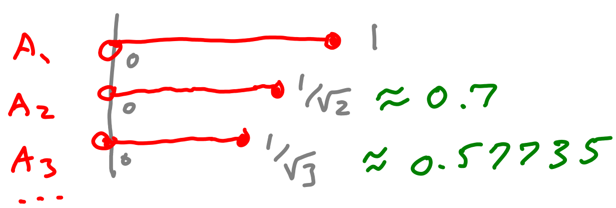
Also $\wp(A) \subseteq \wp(B)$ so $\wp(A) \cup \wp(B) = \wp(B) = \wp(A \cup B)$

4. Suppose A, B, C are sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$



$$[x,y] \in A \times (B \cup C)$$
$$\Leftrightarrow \quad x \in A \wedge y \in B \cup C$$
$$\Leftrightarrow \quad x \in A \wedge (y \in B \vee y \in C)$$
$$\Leftrightarrow \quad (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$
$$\Leftrightarrow \quad [x,y] \in A \times B \;\vee\; [x,y] \in A \times C$$
$$\Leftrightarrow \quad [x,y] \in (A \times B) \cup (A \times C) \quad \ddot\smile$$

5. For each $n \in \mathbf{N}$ let $A_n \subseteq \mathbf{R}$ be the interval $A_n = (0, \frac{1}{\sqrt{n}}]$. Find $\cap\{A_n\colon n \in \mathbf{N}\}$. Prove your assertion.



$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Proof:

If $x \in \bigcap_{n=1}^{\infty} A_n$, $x \in A_1 = (0,1]$, so $x > 0$

By the Archimedean principle

$\exists n \in \mathbb{N} \quad n > \frac{1}{x^2}$. Then $\sqrt{n} > \frac{1}{x}$, so $\frac{1}{\sqrt{n}} < x$

So $x \notin A_n$ $\ddot\frown$