Midterm 1 / 2023.10.4 / MAT 3013.001 / Foundations of Mathematics

1. If $p$ and $q$ are propositions, the contrapositive tautology is that the proposition $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$. Use a truth table to prove this.

```
f(p,q):=[p,q,not p or q,not q, not p,not(not q)or not p];
s:[f(p,q)]$
range:[true,false]$
for p in range
do for q in range
do s:append(s,[f(p,q)])$
apply(matrix,s);
```

(wxmaxima)

$p \Rightarrow q$ $\quad\quad$ $\sim q \Rightarrow \sim p$

$f(p,q) := [p, q, \neg p \vee q, \neg q, \neg p, \neg\neg q \vee \neg p]$

| $p$ | $q$ | $\neg p \vee q$ | $\neg q$ | $\neg p$ | $q \vee \neg p$ |
|---|---|---|---|---|---|
| true | true | true | false | false | true |
| true | false | false | true | false | false |
| false | true | true | false | true | true |
| false | false | true | true | true | true |

2. If $A$ and $B$ are sets, prove that $A \cap B = A$ if and only if $A \subseteq B$

$\Leftarrow$ Suppose $A \subseteq B$

$\subseteq$ $A \cap B \subseteq A$ (if $x \in A \cap B$, $x \in A$)

$\supseteq$ Let $x \in A$. Since $A \subseteq B$, $x \in B$, so $x \in A \cap B$ ◡̈

$\Rightarrow$ Suppose $A \cap B = A$. Let $x \in A$. Then $x \in A \cap B$, so $x \in B$ ◡̈

3. Construct an explicit counterexample using finite sets to the (false) proposition that for any sets $A$ and $B$ we have $\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B)$

Let $A = \{1\}$, $B = \{2\}$, so $A \cup B = \{1, 2\}$.

Then $A \cup B \subseteq A \cup B$, so $A \cup B \in \wp(A \cup B)$, but

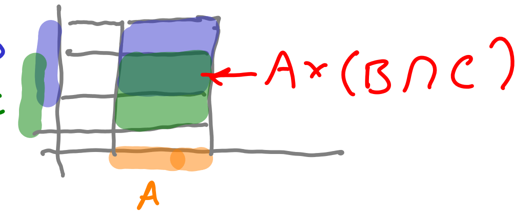
$A \cup B \not\subseteq A$ so $A \cup B \notin \wp(A)$ and $A \cup B \not\subseteq B$, so $A \cup B \notin \wp(B)$,

so $A \cup B \notin \wp(A) \cup \wp(B)$ ◠̈

Note: Suppose $A \subseteq B$. Then $A \cup B = B$.

Also $\wp(A) \subseteq \wp(B)$ so $\wp(A) \cup \wp(B) = \wp(B) = \wp(A \cup B)$

4. Suppose $A, B, C$ are sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$




$[x, y] \in A \times (B \cap C)$

$\iff x \in A \wedge y \in B \cap C$

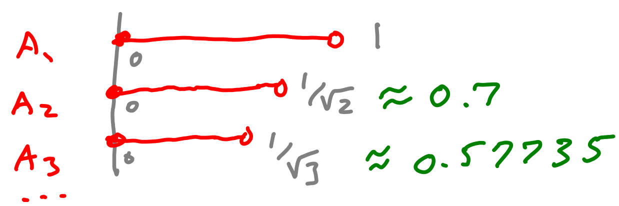
$\iff x \in A \wedge y \in B \wedge y \in C$

$\iff x \in A \wedge y \in B \wedge x \in A \wedge y \in C$

$\iff [x, y] \in A \times B \wedge [x, y] \in A \times C$

$\iff [x, y] \in (A \times B) \cap (A \times C)$ ☺

5. For each $n \in \mathbf{N}$ let $A_n \subseteq \mathbf{R}$ be the interval $A_n = [0, \frac{1}{\sqrt{n}})$. Find $\cap\{A_n : n \in \mathbf{N}\}$. Prove your assertion.




$\bigcap_{n=1}^{\infty} A_n = \{0\}$

Proof: $\supseteq$ $\forall n \in \mathbb{N}$ $0 \in A_n$

$\subseteq$ If $x \in \bigcap_{n=1}^{\infty} A_n$, $x \in A_1 = [0, 1)$, so $x \geq 0$

If $x > 0$, by the Archimedean principle

$\exists n \in \mathbb{N}$ $n > \frac{1}{x^2}$. Then $\sqrt{n} > \frac{1}{x}$, so $\frac{1}{\sqrt{n}} < x$

So $x \notin A_n$ ☹ Thus, $x = 0$, so $x \in \{0\}$