1. Give two different proofs that the sum of all binomial coefficients $\binom{n}{k}$ for a fixed n is 2^n. One using the Binomial theorem and one not.

Binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Plug in $a=b=1$ $\quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$ ☺

2^{nd} proof: $\binom{n}{k}$ = # of subsets of $\{1,\dots n\}$ of size k

$\therefore \sum_{k=0}^{n} \binom{n}{k}$ = total # of subsets of $\{1,\dots n\}$

This $= 2^n$, because a subset of $\{1,\dots n\}$ is determined by whether each element is in the subset or not. That's 2 choices per element.

By the product rule we get $\underbrace{2\cdot 2 \dots 2}_{n} = 2^n$ ☺

2. Prove that the following relations R on a set X are equivalence relations. Sketch several equivalence classes (use different colors for each class). Can you identify the quotient sets X/R?

   (a) $X = \mathbf{N} \times \mathbf{N}$. $[m,n]R[k,l] \Leftrightarrow m + l = k + n$.

   (b) $X = \mathbf{R}$. $xRy \Leftrightarrow x - y \in \mathbf{Z}$.

a) (i) reflexive: $m+n = m+n$ so $[m,n] \, R \, [m,n]$

(ii) Symmetric: by inspection.

(iii) Transitive: If $[m,n] \, R \, [k,l]$ and $[k,l] \, R \, [r,s]$

$m + l = k + n$ and $k + s = r + l$, so

$m + \cancel{l} + s = k + n + s = n + r + \cancel{l}$, so $m + s = r + n$,

So $[m,n] \, R \, [r,s]$.

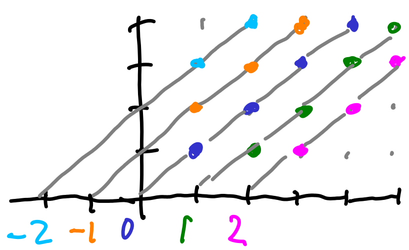
$[1,1] \, R \, [2,2] \, R \, [3,3]$ etc.

$[2,1] \, R \, [3,2] \, R \, [4,3]$ etc.

$[3,1] \, R \, [4,2] \, R \, [5,3]$ etc.

$[1,2] \, R \, [2,3] \, R \, [3,4]$ etc.

$[1,3] \, R \, [2,4]$ etc.



Draw a line through each equivalence class.

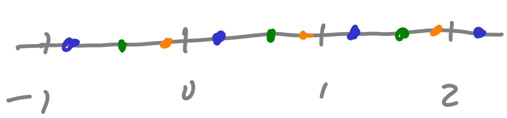
The x intercepts are integers, so $X/_R \cong \mathbb{Z}$

b) (i) reflexive: $\forall r \in \mathbb{R}$ $r - r = 0 \in \mathbb{Z}$

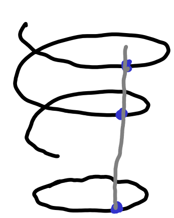
(ii) symmetric: If $r - s \in \mathbb{Z}$, $s - r = -(r-s) \in \mathbb{Z}$

(iii) transitive: If $r - s \in \mathbb{Z}$ and $s - t \in \mathbb{Z}$,

$(r-s) + (s-t) = r - t \in \mathbb{Z}$.

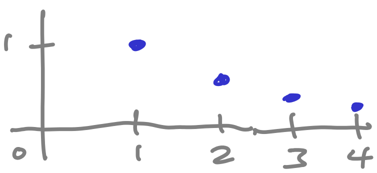


$\therefore X/R$ is a circle



Wrap $\mathbb{R}$ into a helix, each loop of length 1. Draw a vertical line through each eq. class and you get a pt. on a circle.

3. Let $S = \{x \in \mathbf{Q} \colon \exists n \in \mathbf{N}\ x = 1/n\}$ Find $\max S$ and $\min S$ if they exist. Same for $\sup S$ and $\inf S$. Prove your assertions.



(i) $\max S = 1$ (so $\sup S = 1$)

(ii) $\min S$ D.N.E

(iii) $\inf S = 0$

(i) $1 = \frac{1}{1} \in S$. $\forall n \in \mathbb{N}$ $1 \leq n$, so $1 \geq \frac{1}{n}$.

(ii) Let $x \in S$. Then $\exists n$ $x = \frac{1}{n}$. Also $\frac{1}{n+1} \in S$

Since $n+1 > n$, $\frac{1}{n+1} < \frac{1}{n}$, so $\frac{1}{n}$ is not a min

(iii) $\forall n \in \mathbb{N}$ $\frac{1}{n} > 0$, so 0 is a lower bound for S.

Let $r > 0$. By the Archimedian principle,

$\exists n \in \mathbb{N}$ $n > \frac{1}{r}$. Then $\frac{1}{n} \in S$ and $\frac{1}{n} < r$.

$\therefore$ r is not a lower bound for S.

4. Suppose A is a set and $B_k \subseteq A$ for $k \in K$, where K is a nonempty indexing set. Let $S = \{B_k \colon k \in K\} \subseteq \mathscr{P}(A)$. Show that for the partial order $\subseteq$ on $\mathscr{P}(A)$ we have $\sup S = \cup S$ and $\inf S = \cap S$.

$\forall j \in K \quad B_j \subseteq \bigcup_{k \in K} B_k \; (= \cup S)$, so $\cup S$ is an upper bound for S.

Let $T \subseteq \cup S$. If T is an upper bound for S, then

$\forall j \in K \quad B_j \subseteq T$, so $\bigcup_{k \in K} B_k \subseteq T$, so $T = \cup S$

Let $x \in \cup B_k$. Then $\exists j \; x \in B_j$, so $x \in T$

$\therefore \cup S = \sup S$

$\forall j \in K \quad \bigcap_{k \in K} B_k \subseteq B_j$, so $\cap S$ is a lower bound for S.

If P is a lower bound for S, $\forall j \in K \; P \subseteq B_j$

so $P \subseteq \bigcap_{k \in K} B_k$, so $P = \cap S$. :)

Let $x \in P$, then $\forall j \in K \; x \in B_j$, so $x \in \cap S$.