① $n! \leq n^n \qquad n \geq 1$

Basis $(n=1)$: $\quad 1! = 1 \leq 1^1 = 1$ ☺

Let $n > 1$. Assume $k! \leq k^k \quad \forall k < n$

$n! = n\,(n-1)!$

since $n-1 < n \qquad (n-1)! \leq (n-1)^{n-1}$

$n! \leq n\,(n-1)^{n-1} < n \cdot n^{n-1} = n^n$ ☺

(2) $a_0 = 0$, $a_1 = 1$

$a_n = 5a_{n-1} - 6a_{n-2}$ for $n > 1$ $(n \geq 2)$

Prove $a_n = 3^n - 2^n$

Basis $n = 0$: $a_0 = 3^0 - 2^0 = 1 - 1 = 0$ ✓

$n = 1$: $a_1 = 3^1 - 2^1 = 3 - 2 = 1$ ✓

Let $n \geq 2$. Assume $\forall k < n$ $a_k = 3^k - 2^k$

$$a_n = 5a_{n-1} - 6a_{n-2}$$
$$= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2})$$
$$= 5 \cdot 3^{n-1} - 5 \cdot 2^{n-1} - 2 \cdot 3^{n-1} + 3 \cdot 2^{n-1}$$
$$= 3^{n-1}(5-2) + 2^{n-1}(-5+3)$$
$$= 3^{n-1} \cdot 3 + 2^{n-1} \cdot (-2) = 3^n - 2^n$$ ü

(3) $(p \to q) \leftrightarrow (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \leftrightarrow (q \to p)$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Neither

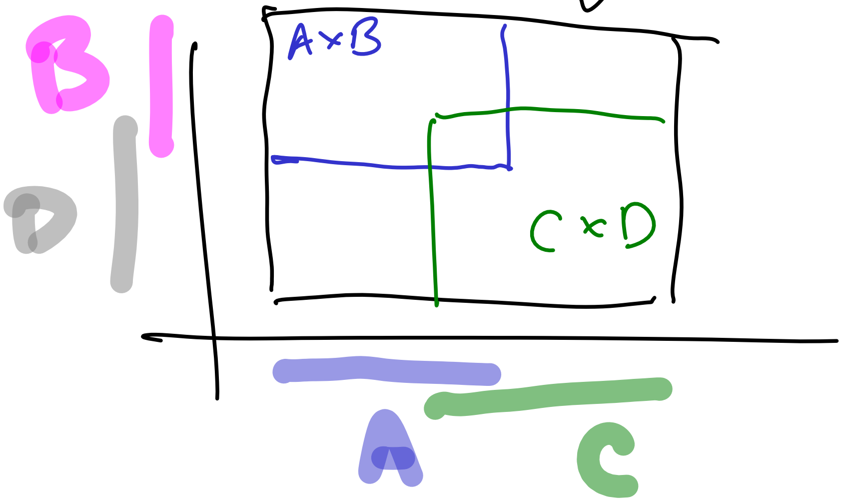
Note: same as $p \leftrightarrow q$

(4) $\{ x \in \mathbb{Z} : 3 \mid x \}$

or $\{ x \in \mathbb{Z} : x \equiv 0 \bmod 3 \}$

or $\{ x \in \mathbb{Z} : \exists k \in \mathbb{Z} \quad x = 3k \}$

5. $(A \cup C) \times (B \cup D) \overset{?}{\subseteq} \overset{?}{\supseteq} (A \times B) \cup (C \times D)$



Claim: $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

Let $[x, y] \in (A \times B) \cup (C \times D)$

$[x, y] \in A \times B \quad \vee \quad [x, y] \in C \times D$

If $[x, y] \in A \times B$, then $x \in A \wedge y \in B$

So $x \in A \cup C$, $y \in B \cup D$

So $[x, y] \in (A \cup C) \times (B \cup D)$

the other case is similar :)

$$(A\times B)\cup(C\times D) \not\supseteq (A\cup C)\times(B\cup D)$$

in general

Counterexample: Let A, B, C, D be distinct singletons $\{a\}, \{b\}, \{c\}, \{d\}$

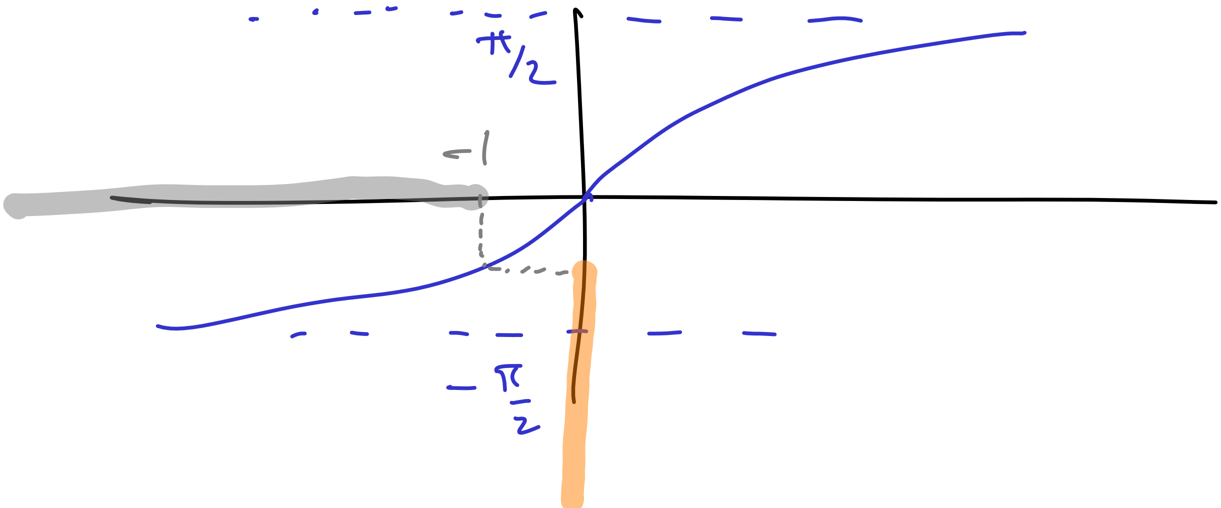
Since $c \in A\cup C$ and $b \in B\cup D$

$[c, b] \in (A\cup C)\times(B\cup D)$

But $c \notin A$, so $[c, b] \notin A\times B$

and $b \notin D$, so $[c, b] \notin C\times D$ ☺

(6) $f : \mathbb{R} \to \mathbb{R}$ $\quad f = \arctan$



a) $f_*(\mathbb{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b) $f^*\left(\left[-\frac{\pi}{4}, -\infty\right)\right) = (-\infty, -1]$

c) Let $A = \{\pi\}$, then $A \neq \emptyset$

But $f^*(A) = \emptyset$.