MAT 3013 Final, Fall 2012

① Basis: $n=1$ $\quad 2^1 3^{2\cdot 1} - 1 = 17$

Also ok: $n=0$ $\quad 2^0 3^{2\cdot 0} - 1 = 0$

Inductive step: Suppose $n > 1$ and $(\exists k \in \mathbb{Z})\left[2^{n-1} 3^{2(n-1)} - 1 = 17k\right]$
(or $n>0$)

Then $2^n 3^{2n} - 1 = 2 \cdot 2^{n-1} 3^2 3^{2n-2} - 1 =$

$= 18 \cdot 2^{n-1} 3^{2(n-1)} - 18 + 18 - 1 = 18\left(2^{n-1} 3^{2(n-1)} - 1\right) + 17 =$

$= 18 \cdot 17k + 17 = 17(18k+1)$

②

p	q	$p \to q$	$q \to p$	$(p\to q) \to (q \to p)$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	1
1	1	1	1	1

Neither a tautology, nor a contradiction

③ Irrationals $= \{x \in \mathbb{R} : \forall m, n \in \mathbb{Z} \quad x \neq \frac{m}{n}\}$

By the axiom schema of separation this is a set.

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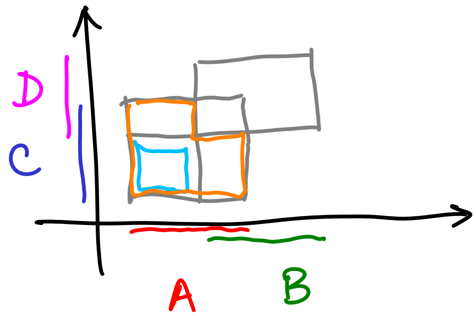


$(A\setminus B)\times(C\setminus D) \subseteq (A\times C)\setminus(B\times D)$

Let $[x,y]\in (A\setminus B)\times(C\setminus D)$

Then $x\in A\setminus B \wedge y\in C\setminus D$

So $x\in A \wedge x\notin B \wedge y\in C \wedge y\notin D$

$x\in A \wedge y\in C \Rightarrow [x,y]\in A\times C$

$x\notin B \Rightarrow [x,y]\notin B\times D$

$\therefore [x,y]\in (A\times C)\setminus(B\times D)$

Counterexample:

Let $A=\{1,2\}$, $B=\{2\}$, $C=\{3,4\}$, $D=\{4\}$

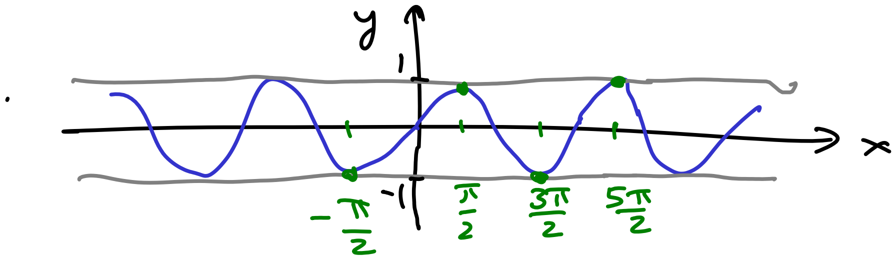
Then $[1,4]\in (A\times C)\setminus(B\times D)$

but $4\notin C\setminus D$ so

$[1,4]\notin (A\setminus B)\times(C\setminus D)$

(5) a) $f_*(\mathbb{R}) = [-1, 1]$

b) $f^*(\{1, -1\}) = \{x \in \mathbb{R} : \exists k \in \mathbb{Z} \quad x = \frac{\pi}{2}(2k+1)\}$



c) Any $A \subseteq \mathbb{R} \setminus f_*(\mathbb{R})$, $A \neq \emptyset$. E.g. $A = \{750\}$

(6) Surjective: $\forall z \in \mathbb{R} \quad z = f([z, 0])$

Not injective: $f([0,0]) = 0 = f([1,1])$

partial inverse: $g : \mathbb{R} \to \mathbb{R}^2 \quad g(x) = [x, 0]$

$f(g(x)) = f([x, 0]) = x$ ☺

Another one: $g'(x) = [x+1, 1]$

$f(g'(x)) = f([x+1, 1]) = x$ ☺

(7) Reflexive: $f(x) = f(x)$, so $x \sim x$

Symmetric: If $x \sim x'$, $f(x) = f(x')$, so $f(x') = f(x)$, so $x' \sim x$

Transitive: If $x \sim x' \wedge x' \sim x''$, then $f(x) = f(x') \wedge f(x') = f(x'')$

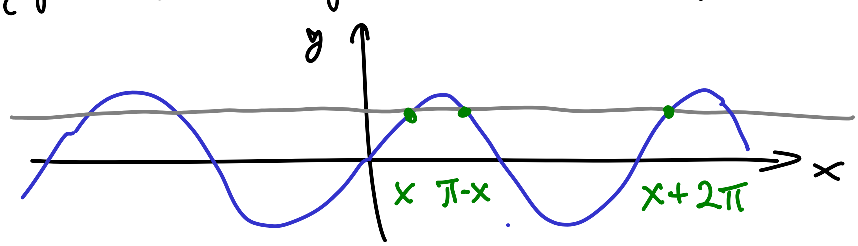
so $f(x) = f(x'')$, so $x \sim x''$

Injective: $x \sim x' \Rightarrow f(x) = f(x') \Rightarrow x = x' \quad \therefore [x] = \{x\}$

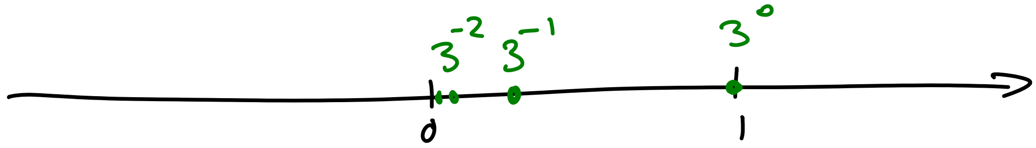
Constant: $\forall x, x' \in X \quad f(x) = f(x')$, so $x \sim x'$, $\therefore [x] = X$

$\sin : \mathbb{R} \to \mathbb{R}$:

$[x] = \{y \in \mathbb{R} : \exists k \in \mathbb{Z} \quad y = x + 2k\pi\} \cup \{y \in \mathbb{R} : \exists k \in \mathbb{Z} \quad y = \pi - x + 2k\pi\}$



(8)



Max $= 1$ (or $\frac{1}{3}$ if you prefer FORTRAN ☺)

No min: Let $x \in S$, then $\exists n \in \mathbb{N}$ $x = 3^{-n}$
Let $y = 3^{-n-1}$. Then $y \in S$ and $y < x$.

Sup $= 1$ (see max)

inf $= 0$: $\forall n \in \mathbb{N}$ $3^{-n} > 0$, so 0 is a lower bound for S.
If $u > 0$, pick $n > -\log_3 u$. Then $-n < \log_3 u$,
So $3^{-n} < u$, so u is not a lower bound for S.

(9) Let D, D' be two Dedekind cuts such that $D \not\subseteq D'$
Then $\exists\, x \in D \setminus D'$. Let $y \in D'$. If $x \leq y$, then $x \in D'$ ☹
$\therefore$ $x > y$, so $y \in D$. ☺

(10) Let $\mathcal{F}$ be a family of initial segments in $\mathbb{Q}$.
Let $x \in \bigcap \mathcal{F}$. Then $\forall A \in \mathcal{F}$ $x \in A$
If $y < x$, $y \in A$, so $y \in \bigcap \mathcal{F}$.

<u>Example</u>: $\bigcap$ all Dedekind cuts $= \emptyset$, so not a Dedekind cut.

<u>Proof</u>: Any rational is not in its own Dedekind cut.
Let $a \in \mathbb{Q}$. $a \notin \{x \in \mathbb{Q} : x < a\}$