① $f: X \to Y$, $A, B \subseteq X$ $f_*(A) \setminus f_*(B) \subseteq f_*(A \setminus B)$

Let $y \in f_*(A) \setminus f_*(B)$

Since $y \in f_*(A)$, $\exists x \in A$ $f(x) = y$

Since $y \notin f_*(B)$, $x \notin B$

$\therefore$ $x \in A \setminus B$, so $y = f(x) \in f_*(A \setminus B)$

Let $X = Y = \mathbb{R}$

Let $f(x) = 0$ $\forall x \in \mathbb{R}$. (f is const.)

Let $A = \mathbb{R}$, $B = \{0\}$

$f_*(A) = \{0\}$, $f_*(B) = \{0\}$

$f_*(A) \setminus f_*(B) = \{0\} \setminus \{0\} = \emptyset$

$f_*(A \setminus B) = \{0\}$ $\not\subseteq$

(2) $f: X \to Y \qquad C, D \subseteq Y$

$C \subseteq D \Rightarrow f^*(C) \subseteq f^*(D)$

Let $x \in f^*(C)$. Then $f(x) \in C$.

Since $C \subseteq D$, $f(x) \in D$, so $x \in f^*(D)$ ü

Let $X = Y = \mathbb{R}$, $f: X \to Y$. $\forall x\ f(x) = 0$

Let $C = \{0, 1\}$, $D = \{0, 5\}$ $(C \not\subseteq D, D \not\subseteq C)$

$f^*(C) = f^*(D) = \mathbb{R}$.

(3) $g \circ f$ is 1-1 $\Rightarrow$ f is 1-1

Pick x, x' such that $f(x) = f(x')$

Then $g(f(x)) = g(f(x'))$

i.e. $(g \circ f)(x) = (g \circ f)(x')$.

But $g \circ f$ is 1-1, so $x = x'$ ☺

Let $X = \mathbb{Z}$, $Y = \mathbb{Q}$, let $f : \mathbb{Z} \to \mathbb{Q}$

$f(n) = \frac{n}{1}$, let $g : \mathbb{Q} \to \mathbb{Z}$ be a rounding function, e.g. given $x \in \mathbb{Q}$ define $g(x)$ as the largest integer $\leq x$.

$g \circ f = Id_{\mathbb{Z}}$

g is not 1-1, eg. $g(0) = g(0.1)$

④ $\mathbb{R}^2 \setminus \{0\}$ $(0,0)$

$[x,y] \sim [x',y'] \iff \exists c \neq 0 \quad x = cx'$
$y = cy'$

Reflexive? Sure. Pick $c=1$.

Symmetric? Suppose $x = cx'$, $y = cy'$ for some $c \neq 0$, Then $x' = \frac{1}{c}x$, $y' = \frac{1}{c}y$ ☺

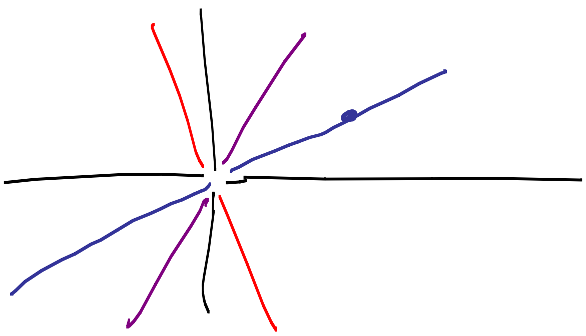
Transitive: Suppose $[x,y] \sim [x',y']$ and $[x',y'] \sim [x'',y'']$.

Then $\exists c \neq 0 \quad x = cx'$, $y = cy'$

$\exists c' \neq 0 \quad x' = c'x'' \quad y' = c'y''$

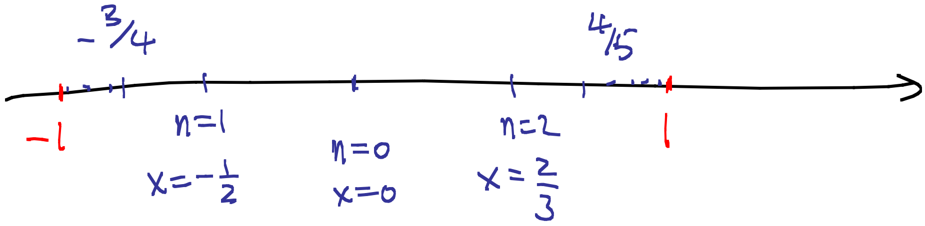
Then $x = cc'x'' \quad y = cc'y''$

Note: $cc' \neq 0$ ☺



(called projective line)

(5) $S = \{ x \in \mathbb{Q} : \exists n \in \mathbb{N} \quad x = \frac{(-1)^n n}{n+1} \}$



No greatest or smallest element.

Suppose m is the smallest element of S.

$\exists k \quad m = \frac{(-1)^k k}{k+1}$ in fact we may assume

$m < 0$, so $m = -\frac{k}{k+1}$ (k is odd)

Then $-\frac{k+2}{k+3}$ is in S and $< -\frac{k}{k+1}$

Check: $\frac{k+2}{k+3} \overset{?}{\underset{>}{}} \frac{k}{k+1}$ $\quad (k+2)(k+1) \overset{>}{?} k(k+3)$

$\underline{\underline{k^2 + 3k + 2}} \underset{>}{?} k^2 + 3k$

$-\frac{k+2}{k+3} < -\frac{k}{k+1}$

$\sup S = 1$, $\inf S = -1$

$-1 < -\frac{k}{k+1}$ $\forall k$ so -1 is a lower bound for S.

Suppose m is a lower bound and $m > -1$.

Since $\frac{k}{k+1} \to 1$ as $k \to \infty$

$\exists\, l$ - odd $1 - \frac{l}{l+1} < m + 1$

$-\frac{l}{l+1} < m$ $\ddot{\frown}$

⑥ If m is the greatest element of A,
it is a upper Bound.
$\forall x \in A$ $M \geq x$

Since m must bound itself, it the least upper bound.