① Prove $n! \geq 2^n$ for $n \geq 4$

$n! = n\underbrace{(n-1)(n-2)(n-3)\cdots 1}_{(n-1)!}$

Basis: $n=4$ $\quad 4! = 24 > 2^4 = 16$ ✓

Inductive Step Assume $(n-1)! \geq 2^{n-1}$

Then $n! = n(n-1)! \geq n \cdot 2^{n-1} \geq 2 \cdot 2^{n-1} = 2^n$ ☺

$n \geq 4 \Rightarrow n \geq 2$

②

p	q	$q \wedge \neg q$	$p \to (q \wedge \neg q)$	$\neg p$	$[p \to (q \wedge \neg q)] \to \neg p$
0	0	0	1	1	1
0	1	0	1	1	1
1	0	0	0	0	1
1	1	0	0	0	1

Alternate solution: **Key:** $p \to q \leftrightarrow \neg p \vee q$

$$p \to (q \wedge \neg q) \leftrightarrow \neg p \vee (q \wedge \neg q)$$

$$(p \to (q \wedge \neg q)) \to \neg p \leftrightarrow \neg(\neg p \vee q \wedge \neg q) \vee \neg p$$

$$\leftrightarrow \big(\underbrace{p \wedge \underbrace{\neg(q \wedge \neg q)}_{\text{false}}}_{\text{true}}\big) \vee \neg p$$

$\neg(q \wedge \neg q) = \neg q \vee q = \text{true}$

$p \wedge \text{true} = p$

$$\leftrightarrow p \vee \neg p = \text{True}$$ ☺

③ $\neg\left[(\exists y)\left[p(y) \wedge (\forall x)\left[\neg q(x) \to r(x) \right]\right] \right]$

$$= (\forall y)\left[\neg p(y) \vee (\exists x)\left[\neg \underbrace{\left(\underbrace{\neg q(x) \to r(x)}_{q(x) \vee r(x)}\right)}_{\neg q(x) \wedge \neg r(x)} \right]\right]$$

$$= (\forall y)\left[\neg p(y) \vee (\exists x)\left[\neg q(x) \wedge \neg r(x) \right]\right] \ddot\smile$$

④ Possible intervals:

$\mathbb{R}$ — done, by assumption

$[a, \infty)$ — $\{x \in \mathbb{R} : x \geq a\}$, so by separation axiom schema, this is a set.

(a, ∞)

(a, b)

$[a, b]$

$(a, b]$ — $\{x \in \mathbb{R} : x > a \wedge x \leq b\}$

$[a, b)$ — The rest are similar

$(-\infty, a]$

$(-\infty, a)$

⑤ $(\forall A)[B \in \wp(A)] \Rightarrow B = \phi$.

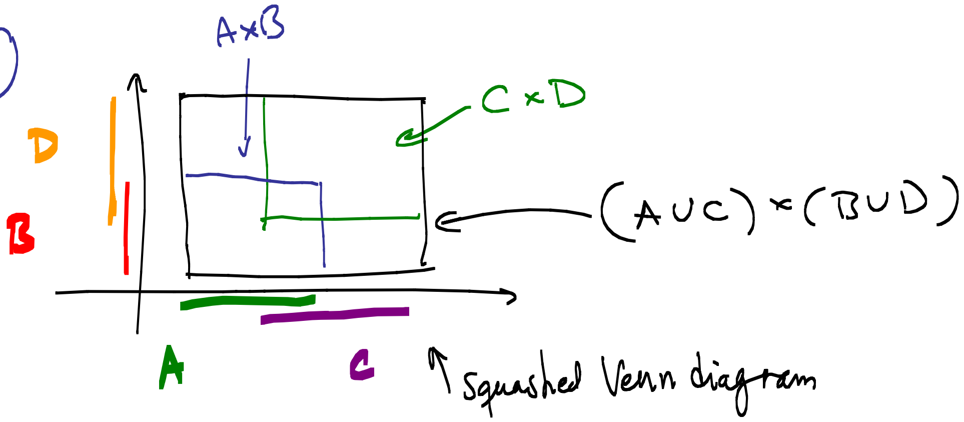
Proof: Suppose $(\forall A)[B \in \wp(A)]$

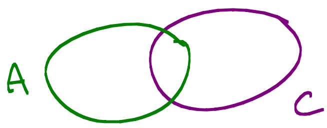
Then if we let $A = \phi$ we get $B \in \wp(\phi)$,

so $B \subseteq \phi$ $\therefore$ $B = \phi$ ☺

(since automatically $\phi \subseteq B$)

⑥





Looks false!

Counterexample: Let $A = \{x_1\}$, $C = \{x_2\}$ where $x_1 \neq x_2$

and $B = \{y_1\}$, $D = \{y_2\}$ where $y_1 \neq y_2$

$$(A \cup C) \times (B \cup D) = \{x_1, x_2\} \times \{y_1, y_2\}$$
$$= \{[x_1, y_1], [x_1, y_2], [x_2, y_1], [x_2, y_2]\}$$

$$(A \times B) \cup (C \times D) = \{[x_1, y_1]\} \cup \{[x_2, y_2]\}$$
$$= \{[x_1, y_1], [x_2, y_2]\}$$

$[x_1, y_2] \in (A \cup C) \times (B \cup D)$, but $\notin (A \times B) \cup (C \times D)$ ☺