1. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$. $r_2 \mapsto r_2 - r_1$, $r_3 \mapsto r_3 - r_1$

   (a) Find a basis for the kernel of A.

   (b) Find a basis for the image of A.

   (c) Describe and sketch the kernel and the image of A.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \quad r_3 \to r_3 - 2r_2 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad r_1 \to r_1 - r_2$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x - z &= 0 \\ y + 2z &= 0 \end{aligned}$$

Pivots

free variable

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

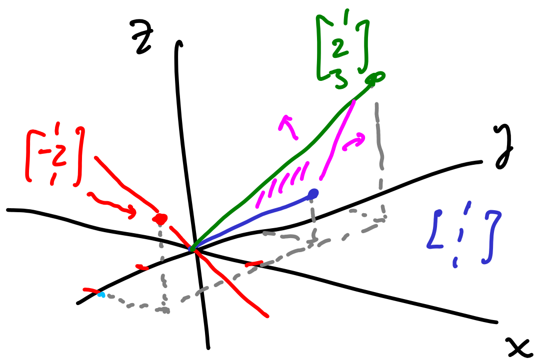
Basis for ker A: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$\text{image}(A) = \text{span}(\text{col's } A) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\text{lin. indep.}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

redundant

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Basis for image(A): $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$



ker(A) = line through the origin

Image(A) = plane through the origin

2. Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Explain why u and v form a basis for the plane $\mathbf{R}^2$.

(b) Express Au as a linear combination of u and v. Same for Av.

(c) What matrix represents the linear map $x \mapsto Ax$ with respect to the u, v basis?

Change of basis matrix: $S = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

a) Since u, v are lin. indep (by inspection) and $\dim \mathbb{R}^2 = 2$ we have a basis.

Another reason: $\det S = -1 \neq 0$, so S is invertible, so u, v is a basis.

$$S^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

c) $$B = S^{-1} A S = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \underbrace{\begin{matrix} 11 \\ 10 \end{matrix}}_{Au} & \underbrace{\begin{matrix} 7 \\ 7 \end{matrix}}_{Av} \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 0 \\ 12 & 7 \end{bmatrix}}$$

b) $$-u + 12v = Au$$
$$7v = Av$$

$$-\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 12 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$$7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

3. Let P be the vector space of polynomials and let $c > 0$. Explain why the following subsets H of P are subspaces of P.

(a) $H = \{p(t) \colon p(c) = 0\}$.

(b) $H = \left\{p(t) \colon \int_0^c p(t)\, dt = 0\right\}$.

a) If q is the zero polynomial ($q(t) = 0$ for all t), then $q(c) = 0$, so q is in H ☺

Given p_1, p_2 in H and c_1, c_2 - real #'s

$p_1(c) = 0, \quad p_2(c) = 0$

$$(c_1 p_1 + c_2 p_2)(c) = c_1 p_1(c) + c_2 p_2(c) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

So $c_1 p_1 + c_2 p_2$ is in H.

$\therefore$ H is a subspace of P. ☺

b) If q is the zero polynomial, $\int_0^c q(t)\,dt = 0$ so q is in H. ☺

Given p_1, p_2 in H and c_1, c_2 - real #'s

$$\int_0^c p_1(t)\,dt = 0, \quad \int_0^c p_2(t)\,dt = 0$$

$$\int_0^c (c_1 p_1 + c_2 p_2)(t)\,dt = \int_0^c [c_1 p_1(t) + c_2 p_2(t)]\,dt$$

$$= c_1 \int_0^c p_1(t)\,dt + c_2 \int_0^c p_2(t)\,dt = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

so $c_1 p_1 + c_2 p_2$ is in H, so H is a subspace of P ☺