1. Let $A = \begin{bmatrix} 21 & 28 \\ -15 & -20 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$.

   (a) Use Gauss-Jordan elimination to compute the reduced row echelon form of the augmented matrix $[A|b]$ and find all solutions to $Ax = b$. Show steps.

   (b) Can you expect some solutions to $Ax = b$ for arbitrary b? Explain.

a) $\left[\begin{array}{cc:c} 21 & 28 & 7 \\ -15 & -20 & -5 \end{array}\right]$ $r_2 \mapsto r_2 + \frac{5}{7} r_1$

$\begin{bmatrix} 21 & 28 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ $r_1 \mapsto \frac{1}{21} r_1$

pivot $\begin{bmatrix} 1 & 4/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}\,[A \,\vdots\, b]$

free variable

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4/3\, y + \frac{1}{3} \\ y \end{bmatrix} = \boxed{y \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}}$$

b) We can't expect a solution for arbitrary b

It depends on whether we get a pivot in the augmentation column in rref([A|b])

$\begin{bmatrix} 1 & 4/3 & * \\ 0 & 0 & * \end{bmatrix}$ if $= 0$ we have solutions

if $\neq 0$ no solutions

2. Assume A and b are as in the above problem.

(a) Express $\ker(A)$ as a span of the least number of vectors by solving $Ax = 0$.

(b) Describe $\ker(A)$ and the solution set to $Ax = b$. Sketch both on the same set of axes.

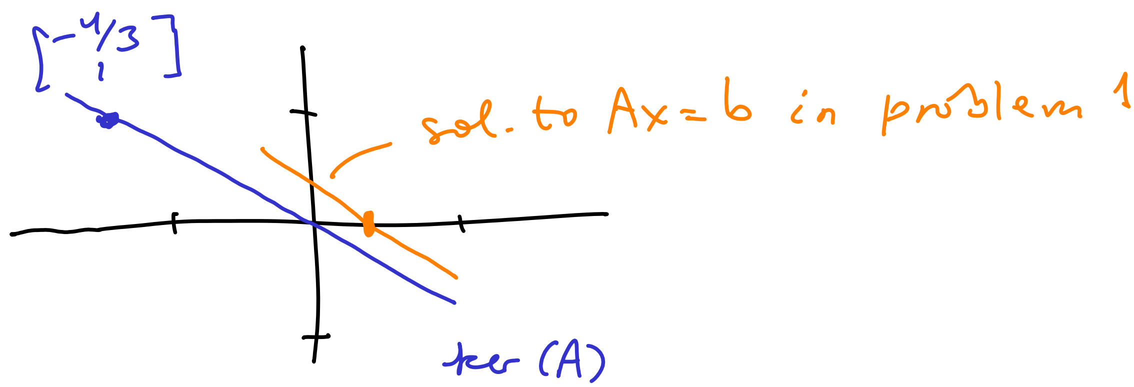
(c) Express the image of $x \mapsto Ax$ as a span of the least number of vectors. Describe and sketch the image.

a) $\text{rref}(A) = \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}$ (from problem 1)

Solution to $A\bar{x} = \bar{0}$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4/3\, y \\ y \end{bmatrix} = \boxed{y \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}}$

$\ker(A) = \text{span} \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$

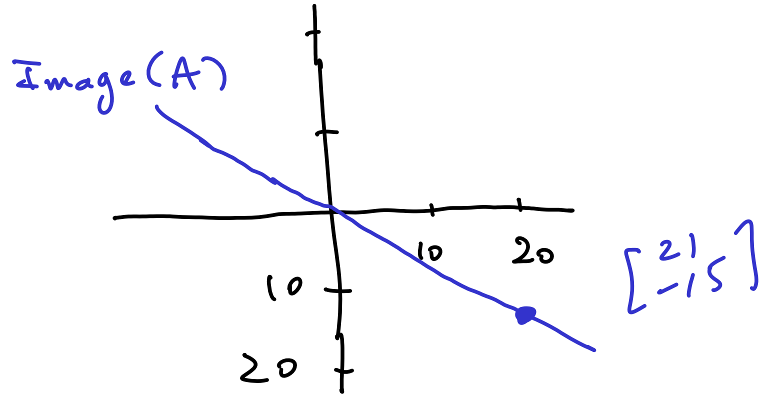
b) Since we have 1 free variable: lines



c) $A\bar{x} = \begin{bmatrix} 21 & 28 \\ -15 & -20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 21 \\ -15 \end{bmatrix} + y \begin{bmatrix} 28 \\ -20 \end{bmatrix}$

$\text{Image}(A) = \text{span}\left(\begin{bmatrix} 21 \\ -15 \end{bmatrix}, \begin{bmatrix} 28 \\ -20 \end{bmatrix} \right)$ ← is a multiple of $\begin{bmatrix} 21 \\ -15 \end{bmatrix}$

$= \text{span} \begin{bmatrix} 21 \\ -15 \end{bmatrix}$



3. Let $u = \begin{bmatrix}1\\2\end{bmatrix}, v = \begin{bmatrix}2\\1\end{bmatrix}$.

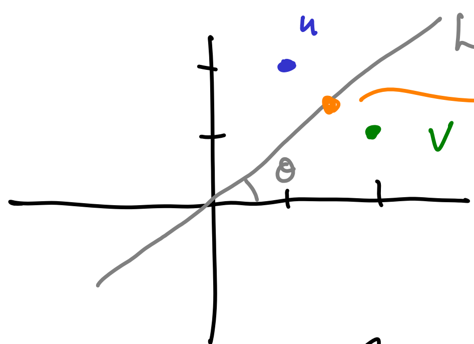
Suppose L is a line through the origin in $\mathbf{R}^2$ and $T : \mathbf{R}^2 \to \mathbf{R}^2$ is the reflection with respect to L. Suppose $T(u) = v$.

(a) Sketch u and v. Then sketch L on the same set of axes.

(b) Find a unit vector $\widehat{u}$ such that $L = \text{span}(\widehat{u})$.

(c) Find a matrix A such that for all vectors x in $\mathbf{R}^2$ we have $T(x) = Ax$.

a)



L = main diagonal

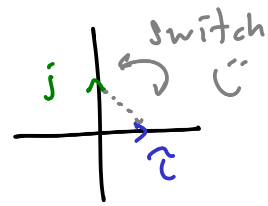
$\begin{bmatrix}3/2\\3/2\end{bmatrix} = \frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}$:)

b) $\theta = \frac{\pi}{4}$ so $\hat{u} = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix} = \boxed{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}}$

Alt: pick any $\bar{u} \neq 0$ on L, e.g. $\bar{u} = \begin{bmatrix}1\\1\end{bmatrix}$, then normalize: $|\bar{u}| = \sqrt{2}$, so $\hat{u} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$:)

c) Since L is the main diagonal

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$



$\therefore$ Matrix $= \boxed{\begin{bmatrix}0&1\\1&0\end{bmatrix}}$

Alt 1 $\quad T(\bar{x}) = 2\,\text{proj}_L \bar{x} - \bar{x}$

$$2\begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix} - \begin{bmatrix}1&0\\0&1\end{bmatrix} = 2\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix} - \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&1\\1&1\end{bmatrix} - \begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}$$

4. Let $u = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, c = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

   Suppose $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a linear map and we know that $T(u) = b$ and $T(v) = c$. Find a matrix A such that for all vectors x in $\mathbf{R}^2$ we have $T(x) = Ax$.

$$T(u) = b \longrightarrow Au = b \qquad A\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$T(v) = c \longrightarrow Av = c \qquad A\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

stack

$$A \underbrace{\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}}_{B}$$

$AS = B$

Solve: $A = BS^{-1}$

$\det S = -1 \quad \therefore \quad S^{-1} = -\begin{bmatrix} -2 & -3 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

Check: $SS^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = BS^{-1} = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 22 \\ 21 & 29 \end{bmatrix}$$

check: $\begin{bmatrix} 16 & 22 \\ 21 & 29 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 16 & 22 \\ 21 & 29 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$