1. Consider the linear system $\begin{bmatrix} 2x+4y+6z=0 \\ 3x+4y+5z=4 \\ 6x+7y+9z=0 \end{bmatrix}$. Use Gauss-Jordan elimination to find all solutions. Show steps. Describe and sketch the solution set. Can you expect some solutions to this system for arbitrary right-hand-sides? Explain.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 4 & 5 & 4 \\ 6 & 7 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -4 & 4 \\ 0 & -5 & -9 & 0 \end{bmatrix}$$

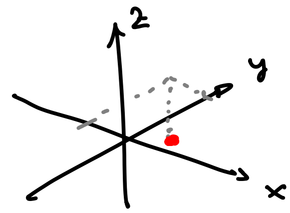
$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & -5 & -9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 30 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

$$\text{rref} = \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

Solution set is a point $\begin{bmatrix} -6 \\ 18 \\ -10 \end{bmatrix}$



3 rows $\Rightarrow$ #pivots ≤ 3

Since each of the 3 coefficient columns (cols. 1-3) has a pivot, augmentation column (4) cannot have a pivot.

$\therefore$ irrespective of r.h.s. the system has a solution

(also a point)

2. Suppose $T\colon \mathbf{R}^2 \to \mathbf{R}$ is a linear map and we know its values at some two (column) vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ that not scalar multiples of one another: $T(\mathbf{u}) = a, T(\mathbf{v}) = b$.

   (a) Let $S = [\mathbf{u}, \mathbf{v}]$. Explain why S is an invertible matrix. What is rref(S)?

   (b) Let A be the matrix that represents T. Let $B = [a, b]$. Explain why $AS = B$.

Hint: Since $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^2$, $A = [A\mathbf{e}_1, A\mathbf{e}_2] = [T(\mathbf{e}_1), T(\mathbf{e}_2)]$, so compute $AS\mathbf{e}_i = T(S\mathbf{e}_i) = \ldots.$

3. Preceding problem continued:

   (c) Use (a) to solve the matrix equation in (b) for A.

   (d) If $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $a = 1$ and $b = -2$ use your solution in (c) to find A.

a) $S = [\bar{u} \;\; \bar{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ (in coords)

If $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\bar{u} = 0 \cdot \bar{v}$ $\quad \therefore \bar{u} \neq 0$

Gauss-Jordan: Swapping rows, if necessary, assume $u_1 \neq 0$.

$$\begin{bmatrix} 1 & v_1/u_1 \\ u_2 & v_2 \end{bmatrix} \to \begin{bmatrix} 1 & v_1/u_1 \\ 0 & v_2 - u_2 \frac{v_1}{u_1} \end{bmatrix}$$

If $v_2 - u_2 \frac{v_1}{u_1} = 0$, $v_2 = u_2 \frac{v_1}{u_1}$, so $\left(v_1 = u_1 \frac{v_1}{u_1}\right)$ $\therefore \bar{v} = \frac{v_1}{u_1} \bar{u}$

$\therefore$ We have 2 pivots $\therefore$ S is invertible $\therefore$ $\text{rref}(S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $A = [A\bar{e}_1 \;\; A\bar{e}_2] = [T(e_1) \;\; T(e_2)]$ $\therefore AS = [AS\bar{e}_1 \;\; AS\bar{e}_2] =$

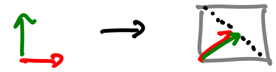
$= [T(S\bar{e}_1) \;\; T(Se_2)] = [T(\bar{u}) \;\; T(\bar{v})] = [a \;\; b] = B$

c) $AS = B \qquad ASS^{-1} = BS^{-1} \qquad A = BS^{-1}$

d) $S = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \qquad B = [1 \;\; -2]$

$S^{-1} = \begin{bmatrix} -3 & 5/2 \\ 2 & -3/2 \end{bmatrix} \qquad \boxed{A = BS^{-1} = \begin{bmatrix} -7 & \frac{11}{2} \end{bmatrix}}$

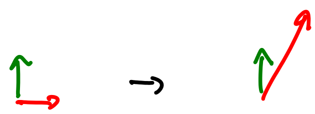
4. In each part enter a real 2×2 nonzero nonidentity matrix A such that the linear map $\mathbf{x} \mapsto A\mathbf{x}$ is as given.

$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ (a) orthogonal projection to the main diagonal 

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) orthogonal reflection with respect to the main diagonal 

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) isotropic dilation 

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (d) rotation by $\frac{\pi}{2}$ 

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (e) vertical shear 

5. Suppose V is an inner product space and U is a subspace of V. Prove that $U^\perp$ is also a subspace of V. Show that any vector in V can be expressed uniquely as a sum of two vectors, one in U and the other in $U^\perp$ (this is the main idea behind Gram-Schmidt).

For any $\bar{u}$ in U, $\langle 0, u\rangle = 0$, so 0 is in $U^\perp$

If $\bar{v}, \bar{w}$ are in $U^\perp$ and a, b are #s, then for any u in $U^\perp$

$$\langle a\bar{v} + b\bar{w}, \bar{u}\rangle = a\underbrace{\langle \bar{v}, \bar{u}\rangle}_{0} + b\underbrace{\langle \bar{w}, \bar{u}\rangle}_{0} = 0$$

$\therefore$ $U^\perp$ is a subspace of V.

Another way to see that $U^\perp$ is a subspace of V is to recognize $U^\perp$ as $\ker P$, where P is the orthogonal projection to U.

Given $\bar{v}$ in V, then $P(\bar{v})$ is in U and

$$P(\bar{v} - P(\bar{v})) = P(\bar{v}) - P(P(\bar{v})) = P(\bar{v}) - P(\bar{v}) = 0$$

$\therefore$ $\bar{v} - P(\bar{v})$ is in $U^\perp$, so $\bar{v} = \underbrace{P(\bar{v})}_{\text{in } U} + \underbrace{\bar{v} - P(\bar{v})}_{\text{in } U^\perp}$.

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If $\bar{v} = \bar{u} + \bar{w}$, where $\bar{u}$ is in U and $\bar{w}$ is in $U^\perp$,

$$P(\bar{v}) = P(\bar{u} + \bar{w}) = P(\bar{u}) + P(\bar{w}) = P(\bar{u}) = \bar{u}$$

and $\bar{w} = \bar{v} - \bar{u} = \bar{v} - P(\bar{v})$, so we have uniqueness.

6. Let P_2 be the vector space of all real polynomials $p(t)$ with degree ≤ 2 and let $\varepsilon\colon P_2 \to \mathbf{R}$ be the evaluation map: $\varepsilon(p(t)) = p(0)$.

(a) Prove that ε is linear. What is the rank of ε? What is the dimension of $\ker \varepsilon$?

(b) Describe $\ker \varepsilon$ and find an orthonormal basis for it relative to the inner product $\langle p(t), q(t)\rangle = \displaystyle\int_0^1 p(t)q(t)\,dt$.

a) If p, q are in P_2 and a, b are #s, then

$$\varepsilon(ap+bq) = (ap+bq)(0) = a\,p(0) + b\,q(0) = a\,\varepsilon(p) + b\,\varepsilon(q)$$

Since ε is not the zero map, $\operatorname{im}(\varepsilon) = \mathbb{R}$ (given a in $\mathbb{R}$, $\varepsilon(a) = a$)

$\therefore$ $\operatorname{rk} \varepsilon = 1$ $\quad\therefore$ $\dim(\ker \varepsilon) = \dim(P_2) - 1 = 3 - 1 = 2$

b) If $p(x) = a_0 + a_1 x + a_2 x^2$, $\quad \varepsilon(p) = p(0) = a_0$, so

$\varepsilon(p) = 0 \iff a_0 = 0$

$\therefore$ $\ker \varepsilon$ is the set of all polynomials in P_2 with 0 const. term.

Let $\bar{v}_1 = t$, $\bar{v}_2 = t^2$. Then $\bar{v}_1, \bar{v}_2$ is a basis for $\ker \varepsilon$.

Gram-Schmidt: $\langle \bar{v}_1, \bar{v}_1 \rangle = \displaystyle\int_0^1 t^2\,dt = \frac{1}{3}$, so $\bar{u}_1 = \dfrac{1}{|\bar{v}_1|}\bar{v}_1 = \boxed{\sqrt{3}\,t}$

$$\bar{v}_2^{\perp} = \bar{v}_2 - \underbrace{\langle \bar{v}_2, \bar{u}_1 \rangle}_{\sqrt{3}\int_0^1 t^3\,dt = \frac{\sqrt{3}}{4}} \bar{u}_1 = t^2 - \frac{3}{4}t$$

$$\langle \bar{v}_2, \bar{v}_2 \rangle = \int_0^1 \left(t^2 - \frac{3}{4}t\right)^2 dt = \int_0^1 \left(t^4 - \frac{3}{2}t^3 + \frac{9}{16}t^2\right) dt = \frac{1}{80},$$

so $\bar{u}_2 = \dfrac{1}{|\bar{v}_2^{\perp}|}\bar{v}_2^{\perp} = \sqrt{80}\left(t^3 - \frac{3}{4}t\right) = \boxed{4\sqrt{5}\,t^2 - 3\sqrt{5}\,t}$

7. Let $A = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 6 & 7 & 2 & 0 \\ 3 & 0 & 0 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix}$ and define $T\colon \mathbf{R}^4 \to \mathbf{R}^4$ by $T(\mathbf{x}) = A\mathbf{x}$. Compute the determinant of A (show work). What can you conclude about T from your answer?

Laplace Expansion (row 1):

$$\det A = 2 \det \begin{bmatrix} 7 & 2 & 0 \\ 0 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} - 4 \det \begin{bmatrix} 6 & 2 & 0 \\ 3 & 0 & 4 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= 2\left(-4 \cdot \det \begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix}\right) - 4\left(6 \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} 3 & 4 \\ 3 & 0 \end{bmatrix}\right)$$

$$= -24 - 4(-24 + 24) = \boxed{-24}$$

Since $\det A \neq 0$, T is invertible.

T expands 4 dimensional content by a factor of 24 and reverses orientation.

8. Let $A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$.

(a) Find the eigenvalues of A and corresponding eigenvectors. Let S be the matrix whose columns are eigenvectors of A. Compute AS. Verify that $S^{-1}AS$ is diagonal with entries the eigenvalues of A.

(b) Sketch the eigenspaces and give a geometrical description of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.

a) $\det(A - \lambda I) = \det \begin{bmatrix} -7-\lambda & 10 \\ -5 & 8-\lambda \end{bmatrix} = (-7-\lambda)(8-\lambda) + 50$

$= \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$, so eigenvals.: $\boxed{\lambda = 3, -2}$

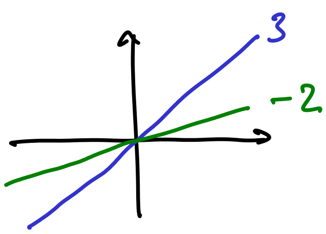
$\mathrm{rref}(A - 3I) = \mathrm{rref}\begin{bmatrix} -10 & 10 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x - y = 0$ so let $\boxed{v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\mathrm{rref}(A + 2I) = \mathrm{rref}\begin{bmatrix} -5 & 10 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $\quad x - 2y = 0$ so let $\boxed{v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

$S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $\quad AS = \begin{bmatrix} 3 & -4 \\ 3 & -2 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, $\quad S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

b)



A dilates by 3 along the main diagonal and dilates by 2 and flips along the line $y = \frac{1}{2}x$