① $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

free

$A\bar{x} = 0$

$\Rightarrow \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

Basis for $\underbrace{\ker}_{\text{plane through } \bar{0} \text{ in } \mathbb{R}^4} A$: $\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$

rank $A = 2$ $\quad \therefore \operatorname{Im} A = \mathbb{R}^2$

② $T : P_3 \to P_3 \qquad T(p) = p'' - kp$

$T(1) = -k$

$T(t) = -kt$

$T(t^2) = 2 - kt^2$

$T(t^3) = 6t - kt^3$

Matrix:

$$\begin{bmatrix} -k & 0 & 2 & 0 \\ 0 & -k & 0 & 6 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -k \end{bmatrix}$$

③ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $AS = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $S^{-1}AS = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = A\bar{v}_1 = -\bar{v}_1 + 4\bar{v}_2 = \begin{bmatrix} -1+4 \\ -1+8 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ ☺

$\begin{bmatrix} 5 \\ 11 \end{bmatrix} = A\bar{v}_2 = -\bar{v}_1 + 6\bar{v}_2 = \begin{bmatrix} -1 & 6 \\ -1 & 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ ☺

④ $\bar{v}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $x + 2y + 3z = 0$

By inspection 1. $\bar{v}_1, \bar{v}_2 \in$ plane

2. $\bar{v}_1, \bar{v}_2$ are linearly indep.

Since dim(plane) = 2, $[\bar{v}_1, \bar{v}_2]$ is a basis.

$|\bar{v}_1| = \sqrt{10}$ $\bar{u}_1 = \begin{bmatrix} -3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{bmatrix}$

$v_2^{\perp} = v_2 - \text{proj}_{\hat{u}_1} v_2 = \bar{v}_2 - \underbrace{(\bar{v}_2 \cdot \hat{u}_1)}_{6/\sqrt{10}} \hat{u}_1$

$= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -18/10 \\ 0 \\ 6/10 \end{bmatrix} \begin{matrix} \leftarrow -9/5 \\ \\ \leftarrow 3/5 \end{matrix} = \begin{bmatrix} -1/5 \\ 1 \\ -3/5 \end{bmatrix}$

$$|v_2^{\perp}| = \sqrt{\frac{1}{25} + 1 + \frac{9}{25}} = \sqrt{\frac{35}{25}} = \sqrt{\frac{7}{5}}$$

$$u_2 = \sqrt{\frac{5}{7}} \begin{bmatrix} -1/5 \\ 1 \\ -3/5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{35}} \\ \sqrt{\frac{5}{7}} \\ -\frac{3}{\sqrt{35}} \end{bmatrix}$$

$$\bar{u}_1 \cdot \bar{v}_2 = \begin{bmatrix} -\frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \frac{6}{\sqrt{10}} = 3\sqrt{\frac{2}{5}}$$

$$Q = \begin{bmatrix} -3/\sqrt{10} & -\frac{1}{\sqrt{35}} \\ 0 & \sqrt{\frac{5}{7}} \\ 1/\sqrt{10} & -\frac{3}{\sqrt{35}} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{10} & 3\sqrt{\frac{2}{5}} \\ 0 & \sqrt{\frac{7}{5}} \end{bmatrix}$$

Check: $QR = \begin{bmatrix} -3 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ ☺