① a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Aug : $\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{array}\right]$

↓ Gauss-Jordan

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} & & \uparrow \text{free} & \end{array}$$

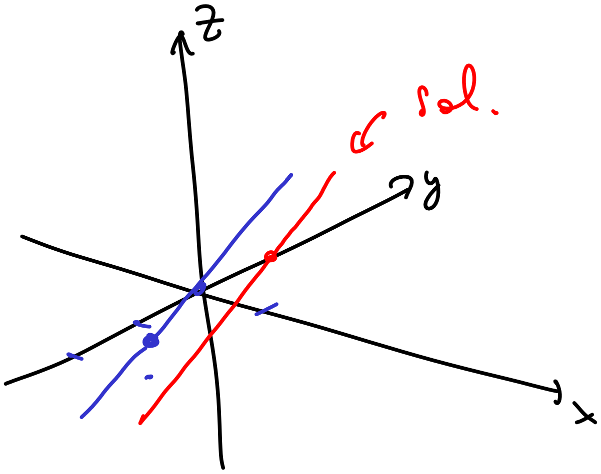
$x_1 = x_3$

$x_2 = -2x_3 + 1$

$$\bar{x} = \begin{bmatrix} x_3 \\ -2x_3+1 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

↑ line in the direction of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ through $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$



b) we have both pivots in the coefficient part, so for any b, no pivot in the augmentation part, so <u>some</u> solutions.

(since rk$(A)=2$, $x \mapsto Ax$ is onto

so for any b, there is x s.t. $Ax=b$)

② $$\left[\begin{array}{cc:c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right]$$

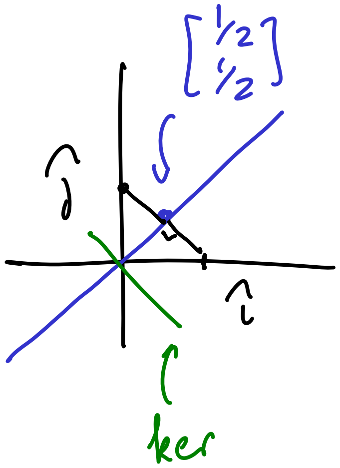
Since the ker is trivial, no free variables, so either unique sol, or no solutions.

b) $$\left[\underbrace{\begin{matrix}1 & 0\\ 0 & 1\\ 0 & 0\end{matrix}}_{A}\ \vdots\ \underbrace{\begin{matrix}1\\ 1\\ 0\end{matrix}}_{b}\right]$$ unique sol

$$\left[\underbrace{\begin{matrix}1 & 0\\ 0 & 1\\ 0 & 0\end{matrix}}_{A}\ \vdots\ \underbrace{\begin{matrix}1\\ 1\\ 1\end{matrix}}_{b}\right]$$ no sol.

③ a) Pick a unit vector along main diagonal

$$\hat{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \therefore \quad \boxed{A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}$$



$$\downarrow \text{G-J}$$

b) $$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \boxed{x = -y}$$

④ $T(2\bar{e}_1 + 3\bar{e}_2) = a$

$T(3\bar{e}_1 + 4\bar{e}_2) = b$

Since T is linear

$2T(\bar{e}_1) + 3T(\bar{e}_2) = a$

$3T(\bar{e}_1) + 4T(\bar{e}_2) = b$

↓ in matrix form

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} T(\bar{e}_1) \\ T(\bar{e}_2) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

↓ Aug

$$\begin{bmatrix} 2 & 3 & a \\ 3 & 4 & b \end{bmatrix}$$

G-J ↓

$$\begin{bmatrix} 1 & 3/2 & a/2 \\ 0 & -1/2 & -\frac{3}{2}a + b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & a/2 \\ 0 & 1 & 3a-2b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4a+3b \\ 0 & 1 & 3a-2b \end{bmatrix}$$

$\therefore$ $T(\bar{e}_1) = -4a+3b$, $T(\bar{e}_2) = 3a-2b$

$$\boxed{A = [-4a+3b \, , \quad 3a-2b]}$$

Alt. $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

$$-4(2\bar{e}_1 + 3\bar{e}_2) + 3(3\bar{e}_1 + 4\bar{e}_2) = \bar{e}_1$$

$$3(2\bar{e}_1 + 3\bar{e}_2) - 2(3\bar{e}_1 + 4\bar{e}_2) = \bar{e}_2$$

$$T(e_1) = T(-4(2\bar{e}_1 + 3\bar{e}_2) + 3(3\bar{e}_1 + 4\bar{e}_2))$$

$$= -4\,T(2\bar{e}_1 + 3\bar{e}_2) + 3\,T(3\bar{e}_1 + 4\bar{e}_2)$$

$$= -4a + 3b$$

Similarly $T(e_2) = 3a - 2b$