MAT 2233.002 Sp '16 final

(1) $\operatorname{rref}\begin{bmatrix}1&1&1&5\\1&2&4&6\\1&3&9&7\end{bmatrix} = \begin{bmatrix}1&0&0&4\\0&1&0&1\\0&0&1&0\end{bmatrix}$ unique solution

A b

$\det(A) = 2$ Thus (or since $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I$) A is invertible, so for any b, $x = A^{-1}b$

(2) $\operatorname{rref}\begin{bmatrix}1&2&5&6\\1&1&7&8\end{bmatrix} = \begin{bmatrix}1&0&9&10\\0&1&-2&-2\end{bmatrix}$ $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-9\\2\\1\\0\end{bmatrix}x_3 + \begin{bmatrix}-10\\2\\0\\1\end{bmatrix}x_4$

Image is all of $\mathbb{R}^2$ free

Basis for Kernel

(3) Rotation: $\hat{i} \mapsto -\hat{j}$, $\hat{j} \mapsto \hat{i}$ so $A = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$

$A^2 = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}$, so $A^4 = (A^2)^2 = I$ ← rotation by $-360°$ ☺

Reflection: $\hat{i} \mapsto \hat{j}$, $\hat{j} \mapsto \hat{i}$, so $A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$. $A^2 = I$ (so $A^4 = I$)

even # of flips ☺

(4) $T(1) = 1-1 = 0$, $T(t) = t+1-(t-1) = 2$, $T(t^2) = (t+1)^2 - (t-1)^2 = 4t$ $\therefore$ $A = \boxed{\begin{bmatrix}0&2&0\\0&0&4\\0&0&0\end{bmatrix}}$

(5) Let $S = \begin{bmatrix}2&1\\1&1\end{bmatrix}$, then $S^{-1} = \begin{bmatrix}1&-1\\-1&2\end{bmatrix}$, $S^{-1}AS = \begin{bmatrix}1&-1\\-1&2\end{bmatrix}\underbrace{\begin{bmatrix}3&4\\5&6\end{bmatrix}\begin{bmatrix}2&1\\1&1\end{bmatrix}}$ $\begin{bmatrix}10&7\\16&11\end{bmatrix}$

$= \underline{\begin{bmatrix}-6&-4\\22&15\end{bmatrix}}$ $\therefore$ $Av_1 = \underline{-6v_1 + 22v_2}$, $Av_2 = \underline{-4v_1 + 15v_2}$ (check ☺)

(6) $1-2+1=0$, $-1+0+1=0$ ☺ Let $v_1 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$. Then $v_1 \cdot v_2 = 0$.

$|v_1| = \sqrt{6}$, $|v_2| = \sqrt{2}$ $\therefore$ $Q = \boxed{\begin{bmatrix}1/\sqrt{6} & -1/\sqrt{2}\\ -2/\sqrt{6} & 0\\ 1/\sqrt{6} & 1/\sqrt{2}\end{bmatrix}}$, $R = \boxed{\begin{bmatrix}\sqrt{6} & 0\\ 0 & \sqrt{2}\end{bmatrix}}$

(7) $A^tA = \begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix} = \begin{bmatrix}2&1\\1&2\end{bmatrix}$, $A^tb = \begin{bmatrix}6\\8\end{bmatrix}$, solve $A^tAx^* = A^tb$: $\boxed{x^* = \begin{bmatrix}4\\-2\end{bmatrix}}$

$b - Ax^* = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$ ← perp to $\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ ← basis for image of A.

(8) $A = \begin{bmatrix} -9 & 8 \\ -12 & 11 \end{bmatrix}$ $\det(A - \lambda I) = \det\begin{bmatrix} -9-\lambda & 8 \\ -12 & 11-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

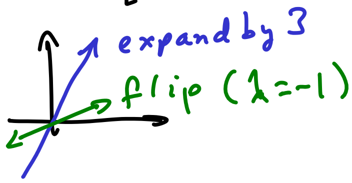
$\lambda = 3, -1$

$\text{rref}(A - 3I) = \text{rref}\begin{bmatrix} -12 & 8 \\ -12 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$ $\quad x - \frac{2}{3}y = 0 \quad v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

eigenvectors

$\text{rref}(A + I) = \text{rref}\begin{bmatrix} -8 & 8 \\ -12 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x - y = 0 \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $S = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$, then $S^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$, $A \cdot S = \begin{bmatrix} 6 & -1 \\ 9 & -1 \end{bmatrix}$, $\underline{S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}$



(9) Let $A = [a_{ij}]$ be upper triangular (i.e. $i > j \Rightarrow a_{ij} = 0$)

Since any nontrivial permutation introduces a 0 entry to the diagonal (If it moves the 1st col. we get a 0. If not, but the 2nd moves, it must move to the right, so again we get a 0, etc.), only the main diagonal contributes to the determinant, so $\det A = a_{11}a_{22}\cdots a_{nn}$

Another option: Induction on n. $[a]$ ✓ Let $n > 1$ and do a Laplace expansion along n^{th} row: the M_{nn} minor is $(n-1)\times(n-1)$ and upper triangular, so $\det A = \det M_{nn} \cdot a_{nn}(-1)^{n+n} = (a_{11}\cdots a_{n-1,n-1}) \cdot a_{nn}$

If A is u.t., so is $A - \lambda I$, so $\det(A - \lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$:)