Midterm 1 2007.3.8 Calculus III MAT2213.003

① a) $0.6 + 0.36 + 0.216 + \ldots = \sum_{n=1}^{\infty} 0.6^n$

$$= 0.6 \sum_{n=1}^{\infty} 0.6^{n-1} = 0.6 \cdot \sum_{n=0}^{\infty} 0.6^n = 0.6 \frac{1}{1-0.6}$$

$$= \frac{0.6}{0.4} = \boxed{\frac{3}{2}}$$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x|<1$

b) $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1} = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} \ldots\right]$

etc.

Since the terms go to 0 as $n \to \infty$ the sum converges to $\frac{1}{2}\left[1 + \frac{1}{2}\right] = \boxed{\frac{3}{4}}$

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② a) $\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \ldots = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Try integral test:

$\int_2^{\infty} \frac{1}{x \ln x}\,dx$ Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$

$$\int_{\ln 2}^{\infty} \frac{du}{u} = \ln u \Big|_{\ln 2}^{\infty} = \underbrace{\lim_{u\to\infty} \ln u}_{\to \infty} - \ln(\ln 2)$$

$\therefore$ the series diverges

b) $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ Compare to $\sum \frac{n}{(n^2)^2} = \sum \frac{1}{n^3}$

converges by the p-test with $p=3>1$

$$\frac{5+2n}{(1+n^2)^2} \Big/ \frac{1}{n^3} = \frac{(5+2n)n^3}{(1+n^2)^2}$$

$$= \frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} \to 2$$

($\frac{5}{n} \to 0$, $\frac{1}{n^2} \to 0$)

Since $2 \neq 0$ & $2 \neq \infty$ the comparison is valid & the original series converges.

(3) $\frac{1}{4+x^2} = \frac{1}{4}\frac{1}{1+\frac{x^2}{4}} =$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n} = \frac{1}{4} - \frac{1}{16}x^2 + \frac{1}{64}x^4 \ldots$$

Convergence? Ratio test:

$$\left|\frac{(-1)^{n+1}}{4^{(n+1)+1}}x^{2(n+1)} \Big/ \frac{(-1)^n}{4^{n+1}}x^{2n}\right| = \frac{1}{4}|x|^2$$

$$\frac{1}{4}|x|^2 < 1 \iff |x|^2 < 4 \iff |x| < 2$$

$\therefore$ radius of convergence is 2.

Endpoints? $x = 2$: $\sum \frac{(-1)^n}{4^{n+1}} 2^{2n} = \sum \frac{(-1)^n}{4^{n+1}} 4^n = \frac{1}{4}\sum(-1)^n$

diverges since the terms $\not\to 0$.

$x = -2$ same thing because $(-2)^{2n} = 2^{2n}$

$\therefore$ the interval of convergence is $\boxed{-2 < x < 2}$

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(4) The angle of $\begin{bmatrix}5\\4\end{bmatrix}$ is $\arctan\left(\frac{4}{5}\right) = 0.6747$ $(38.66°)$

We want that $\pm\frac{\pi}{6}$:

$\theta_1 = 0.15$ $(8.66°)$, $\theta_2 \simeq 1.2$ $(68.66°)$

$$\begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix} = \begin{bmatrix}0.9886\\ 0.15\end{bmatrix}, \quad \begin{bmatrix}\cos\theta_2\\ \sin\theta_2\end{bmatrix} \simeq \begin{bmatrix}0.3639\\ 0.93\end{bmatrix}$$

Alternate method:

Suppose $v = \begin{bmatrix}x\\y\end{bmatrix}$, $|v| = 1$ and the angle with $u = \begin{bmatrix}5\\4\end{bmatrix}$ is $\frac{\pi}{6}$

then $x^2 + y^2 = 1$

and $u \cdot v = 5x + 4y = |u|\cdot|v|\cos\frac{\pi}{6} = \sqrt{5^2+4^2}\cdot 1 \cdot \frac{\sqrt{3}}{2}$

$$= \frac{\sqrt{41}\sqrt{3}}{2} = \frac{\sqrt{123}}{2}$$

Solve for y : $y = \frac{1}{4}\left(\frac{\sqrt{123}}{2} - 5x\right)$

$$\therefore\ x^2+y^2 = x^2 + \left[\frac{1}{4}\left(\frac{\sqrt{123}}{2} - 5x\right)\right]^2 = 1$$

$$x^2 + \frac{1}{16}\left(\frac{123}{4} - 5\sqrt{123}\,x + 25x^2\right) = 1$$

$$\left(1 + \frac{25}{16}\right)x^2 - \frac{5\sqrt{123}}{16}x + \frac{123}{16\cdot 4} - 1 = 0$$

$$2.5625\,x^2 - 3.46579\,x + 0.921875 = 0$$

$$x^2 - 1.3525\,x + 0.359756 = 0$$

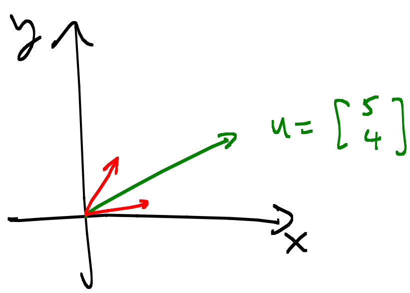
$$x = \frac{1.3525 \pm \sqrt{1.3525^2 - 4\cdot 0.359756}}{2}$$

$$= 0.9886\ ,\ 0.3639$$

If $x = 0.9886$, then $y = \frac{1}{4}\left(\frac{\sqrt{123}}{2} - 5\cdot 0.9886\right) = 0.15$

If $x = 0.3639$, then $y = \frac{1}{4}\left(\frac{\sqrt{123}}{2} - 5\cdot 0.3639\right) = 0.93$

$$\therefore\ v = \begin{bmatrix} 0.9886 \\ 0.15 \end{bmatrix},\ \begin{bmatrix} 0.3639 \\ 0.93 \end{bmatrix}$$



5 Suppose $A = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$, $C = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

Let $u = B - A = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$

$v = C - A = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$

Since u & v are parallel to the plane,

$$u \times v = \begin{array}{c} \begin{bmatrix} 4 & 5 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} \\ 4 \quad 5 \end{array} = \begin{bmatrix} 13 \\ -14 \\ 5 \end{bmatrix}$$ is $\perp$ to the plane

$$|u \times v| = \sqrt{13^2 + 14^2 + 5^2} = \sqrt{390}$$

$\therefore$ Area of triangle $= \frac{1}{2}\sqrt{390} = \underline{9.8742}$

Normalize: $$\frac{u \times v}{|u \times v|} = \frac{1}{\sqrt{390}} \begin{bmatrix} 13 \\ -14 \\ 5 \end{bmatrix} = \underline{\begin{bmatrix} 0.66 \\ -0.71 \\ 0.25 \end{bmatrix}}$$ $\leftarrow$ unit vector $\perp$ to the plane