Calculus III, MAT 2213.002 Exam, Oct. 18, 1993. Instructor: D. Gokhman

Show all pertinent work, answers alone are not sufficient. Box the answers.

Name: ______________________________

1. (30 pts.) Find the interval of convergence for the following power series:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}} \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{(-3)^{n+1}(x-5)^n}{2^{3n}}$$

2. (30 pts.) Determine whether each of the following sequences or series converges to a real number.

$$\text{(a)} \frac{\sqrt{n}\ln(n)}{n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}\cos(n\pi)}{(n^5+1)^{\frac{1}{4}}}$$

3. (20 pts.) Find the Taylor polynomial for $\sqrt[5]{x}$ of degree $n = 2$ centered at $a = 32$. Estimate the error of approximating $\sqrt[5]{30}$ with the above polynomial.

4. (20 pts.) For each of the following functions $f(x)$ find the Taylor series with center $a = 0$.

$$\text{(a) } f(x) = \frac{x^{10}}{5-x^2} \qquad \text{(b) } f(x) = \frac{x}{(1+x)^2}$$

1	2	3	4	total (100)

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1) a) Root test $|a_n|^{\frac{1}{n}} = \sqrt[n]{\sqrt{n}} =$

$= \sqrt{\sqrt[n]{n}} \to 1 \qquad R = 1/1 = 1$

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Endpoints: $-2-1, \; -2+1$

3 $x = -3$: $\sum \frac{(-1)^n}{\sqrt{n}}$ Conv. by Alt. Series test. Since $\frac{1}{\sqrt{n}} \to 0$ monotonically

3 $x = -1$ $\sum \frac{1}{\sqrt{n}}$ div as a p-series with $p = \frac{1}{2} < 1$

$$\boxed{[-3, -1)}$$

b) $\sum \frac{(-3)^{n+1}(x-5)^n}{2^{3n}}$

Root test: $|a_n|^{\frac{1}{n}} = \frac{|-3|^{\frac{n+1}{n}}}{2^3} \to \frac{3}{2^3} = \frac{3}{8}$

$R = \frac{1}{3/8} = \frac{8}{3}$

Endpoints: $5 - \frac{8}{3}, \; 5 + \frac{8}{3}$

$x = 5 + \frac{8}{3}$: $\sum \frac{(-1)^{n+1} 3^{n+1}}{2^{3n}} \frac{8^n}{3^n} = \sum (-1)^{n+1} 3$ div. since $3 \not\to 0$

$x = 5 - \frac{8}{3}$: $\sum \frac{(-1)^{n+1} 3^{n+1}}{2^{3n}} \frac{(-1)^n 8^n}{3^n} = \sum -3$ div.

$$\boxed{\left(5 - \frac{8}{3}, 5 + \frac{8}{3}\right)}$$

Ans = 3p

2a) $\frac{\sqrt{n}\ \ln n}{n} = \frac{\ln n}{\sqrt{n}} \to 0.$ $\boxed{\text{conv}}$

Check with l'Hôpital's rule: $\frac{1/n}{\frac{1}{2\sqrt{n}}} = \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}} \to 0$

b) Ratio $\frac{|a_{n+1}|}{|a_n|} = \frac{(2(n+1))!}{(3(n+1))!} \Big/ \frac{(2n)!}{(3n)!} =$

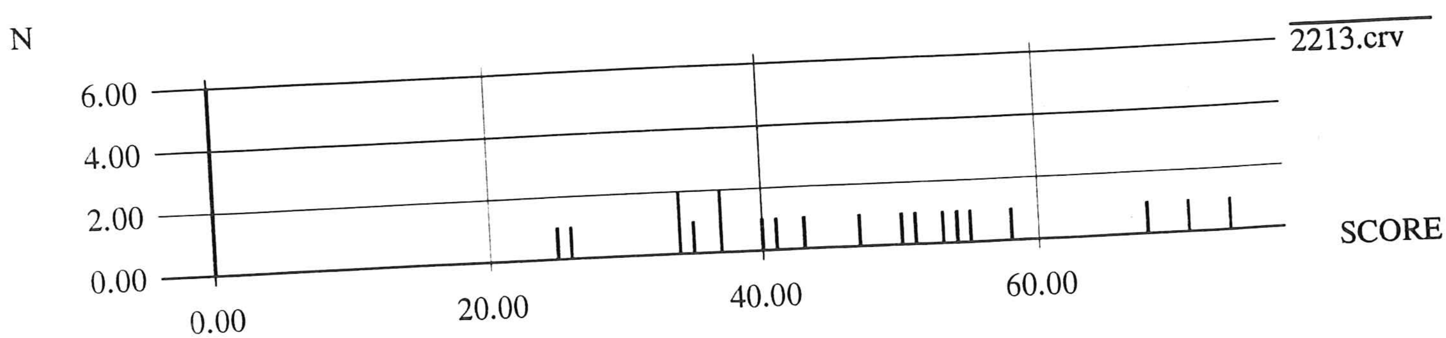
$= \frac{(3n)!}{(3n+3)!} \frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \approx \frac{2^2 n^2}{3^3 n^3} \to 0 < 1$

$\boxed{\text{Conv}}$

3) $\sum \frac{n^{\frac{1}{2}}(-1)^n}{(n^5+1)^{1/4}}$ $\boxed{\text{Conv}}$ by alternating series test.

since $\frac{n^{\frac{1}{2}}}{(n^5+1)^{1/4}} \approx \frac{n^{1/2}}{n^{5/4}} = n^{\frac{1}{2}-\frac{5}{4}} = n^{-\frac{3}{4}} \to 0$

2213 testsc[0]



(3) $f(x) = x^{\frac{1}{5}}$

$f(32) = 2 \qquad a_0 = 2$

$f'(x) = \frac{1}{5} x^{-\frac{4}{5}} \qquad f'(32) = \frac{1}{5 \cdot 2^4} \qquad a_1 = \frac{1}{5 \cdot 2^4}$

$f''(x) = \frac{1}{5}\left(-\frac{4}{5}\right) x^{-\frac{9}{5}} \qquad f''(32) = -\frac{4}{5^2 2^9} = -\frac{1}{5^2 2^7}$

$f'''(x) = \frac{1}{5}\left(\frac{-4}{5}\right)\left(-\frac{9}{5}\right) x^{-\frac{14}{5}} = \frac{4 \cdot 9}{5^3} x^{-\frac{14}{5}}$

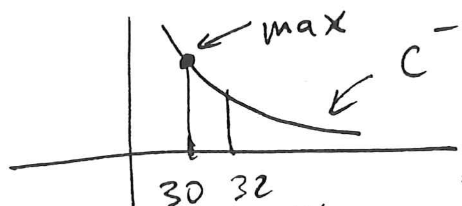
↑ divide by 2

$a_2 = -\frac{1}{5^2 2^8}$

$$\boxed{P_2 = 2 + \frac{1}{5 \cdot 2^4}(x-32) - \frac{1}{5^2 2^8}(x-32)^2}$$

$R_2 = \frac{f'''(c)}{3!}(x-32)^3 = \frac{4 \cdot 9}{5^3 \cdot 3!} c^{-\frac{14}{5}} (x-32)^3$

$\underset{[30,32]}{\text{Max}}\ c^{-\frac{14}{5}} = 30^{-\frac{14}{5}}$



$|R_2| \leq \frac{6}{5^3} 30^{-\frac{14}{5}} |30-32|^3 = \frac{6}{5^3} 30^{-\frac{14}{5}} 2^3$

$$\boxed{|R_2| \leq \frac{6 \cdot 2^3}{5^3} 30^{-\frac{14}{5}}}$$

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(4)

a) $$\frac{x^{10}}{5-x^2} = \frac{x^{10}}{5}\left(\frac{1}{1-\frac{x^2}{5}}\right) =$$

$$= \frac{x^{10}}{5}\left(1 + \frac{x^2}{5} + \left(\frac{x^2}{5}\right)^2 + \left(\frac{x^2}{5}\right)^3 + \dots + \left(\frac{x^2}{5}\right)^k + \dots\right)$$

$$= \frac{x^{10}}{5}\left(1 + \frac{x^2}{5} + \frac{x^4}{5^2} + \frac{x^6}{5^3} + \dots + \frac{x^{2k}}{5^k} + \dots\right) =$$

$$\boxed{= \frac{x^{10}}{5} + \frac{x^{12}}{5^2} + \frac{x^{14}}{5^3} + \dots + \frac{x^{2k+10}}{5^{k+1}} + \dots}$$

b) $$\frac{x}{(1+x)^2} = x \cdot \left(-\frac{d}{dx}\left(\frac{1}{1+x}\right)\right) =$$

$$= x \cdot \left[-\frac{d}{dx}\left(1 - x + x^2 - x^3 + \dots + (-1)^k x^k + \dots\right)\right] =$$

$$= -x\left[-1 + 2x - 3x^2 + \dots + (-1)^k k x^{k-1} + \dots\right] =$$

$$\boxed{= +x - 2x^2 + 3x^3 - \dots + (-1)^{k+1} k x^k + \dots}$$