Calculus III, MAT 2213.001. Exam, Oct. 18, 1993. Instructor: D. Gokhman

Show all pertinent work, answers alone are not sufficient. Box the answers.

Name: ______________________________

1. (30 pts.) Find the interval of convergence for the following power series:

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{(n-2)(x-1)^n}{n^2} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{x^n}{3^n n^2}$$

2. (30 pts.) Determine whether each of the following sequences or series converges to a real number. If so, find the limit. Otherwise state that the sequence or series diverges.

$$\text{(a)}\ \left(\frac{n-1}{n}\right)^{(n^2)} \qquad \text{(b)}\ \cos\left(\frac{n\pi}{4}\right) \qquad \text{(c)}\ \sum_{n=0}^{\infty} \left(\frac{1}{5^n} + \frac{2}{3^{n+1}}\right)$$

3. (20 pts.) For each of the following series find the set of all p such that the given series converges.

$$\text{(a)}\ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^p} \qquad \text{(b)}\ \sum_{n=0}^{\infty} \frac{1}{n\,(\ln n)^p}$$

4. (20 pts.) Find the Taylor polynomial for $\ln x$ of degree $n = 2$ centered at $a = 1$. Estimate the error of approximating $\ln(9/10)$ with the above polynomial.

1	2	3	4	total (100)

8

1) a) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)-2}{(n+1)^2} \Big/ \dfrac{n-2}{n^2} = \dfrac{(n-1)\cdot n^2}{(n-2)(n+1)^2} \to 1$ /5

$R = 1/1 = 1$

Endpoints: $1-1=0$, $1+1=2$

$x=0 \to \sum \dfrac{(n-2)(-1)^n}{n^2}$ conv. by Alt. Series test since $\dfrac{n-1}{n^2} \approx \dfrac{1}{n} \to 0$

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$x=2$ $\quad \sum \dfrac{n-2}{n^2}$ div. by comparison to p-series $\dfrac{n-2}{n^2} \approx \dfrac{1}{n}$ $\quad p=1$.

$\boxed{\text{Interval: } [0,2)}$ 5

b) $\sum \dfrac{x^n}{3^n n^2}$

$|a_n|^{\frac{1}{n}} = \dfrac{1}{3\, n^{2/n}} \to \dfrac{1}{3}$ $\quad R = 3$

Endpoints: $0-3$, $0+3$

$x=-3$ $\quad \cancel{\dfrac{(-1)^n 3^n}{3^n n^2}}$ $\sum \dfrac{(-3)^n}{3^n n^2} = \sum \dfrac{(-1)^n}{n^2}$ conv.

$x=3$ $\quad \sum \dfrac{3^n}{3^n n^2} = \sum \dfrac{1}{n^2}$ conv.

$\boxed{\text{Interval: } [-3,3]}$

(2) a) $\left(\frac{n-1}{n}\right)^{(n^2)} = \left(1-\frac{1}{n}\right)^{n\cdot n} = \left[\left(1-\frac{1}{n}\right)^n\right]^n =$

$= \longrightarrow [e^{-1}]^n = e^{-n} \longrightarrow \boxed{0.}$ -10

b) $\cos\left(\frac{n\pi}{4}\right) = \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \ldots$

$n = 1, 2, 3, 4, 5, 6, 7, 8$

$\boxed{\text{diverges.}}$ (5) -10

c) $\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$

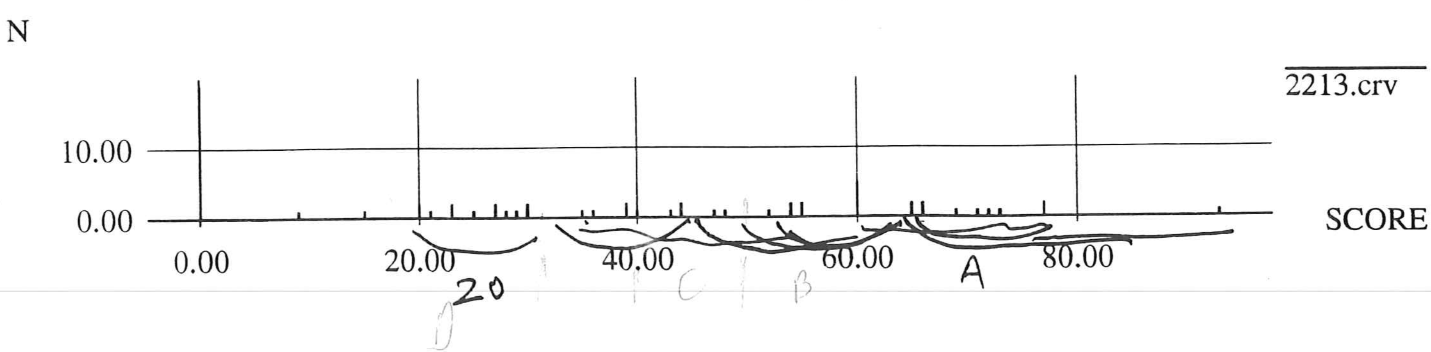
$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{1}{2/3} = \frac{3}{2}$

$\sum\left(\frac{1}{5^n} + \frac{2}{3^{n+1}}\right) = \sum \frac{1}{5^n} + \frac{2}{3}\sum\frac{1}{3^n} =$ -10

$= \frac{5}{4} + \frac{2}{3}\cdot\frac{3}{2} = \frac{5}{4} + 1 = \boxed{\frac{9}{4}}$

(5)

2213 testsc[0]



3) a) If $p > 0$ $\frac{1}{n^p} \to 0$ monotonically

so by alt. series test the series converges

If $p \le 0$ $\frac{1}{n^p} \to \begin{cases} 1 & \text{if } p = 0 \\ \infty & \text{if } p < 0 \end{cases}$

so By the k^{th} term test the series diverges

$\boxed{\text{Ans: } p > 0}$

b) $\sum \frac{1}{n(\ln n)^p}$

Integral test: $\int \frac{1}{x(\ln x)^p} dx$

let $u = \ln x$ $du = \frac{1}{x} dx$

$\int \frac{1}{u^p} du = \int u^{-p} du = \frac{u^{-p+1}}{-p+1}$ if $p \neq 1$.

As $x \to \infty$, $u \to \infty$.

As $u \to \infty$ $u^{-p+1} \to \begin{cases} \infty & \text{if } -p+1 > 0 \\ 0 & \text{if } -p+1 < 0 \end{cases}$

So conv. for $p > 1$

div. for $p < 1$

If $p = 1$ $\int \frac{1}{u} du = \ln u$ — diverges.

$\boxed{\text{Ans: } p > 1}$

(4) $f(x) = \ln x$

$f(1) = 0 \qquad a_0 = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1 \qquad a_1 = 1$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1 \qquad a_2 = -\frac{1}{2}$

$f'''(x) = \frac{2}{x^3}$

$$\boxed{P_2 = (x-1) - \frac{1}{2}(x-1)^2} \qquad 10$$

$$R_2 = \frac{f'''(c)}{3!}(x-1)^3 = \frac{2}{3!\,c^3}(x-1)^3 = \frac{1}{3}\frac{1}{c^3}(x-1)^3$$

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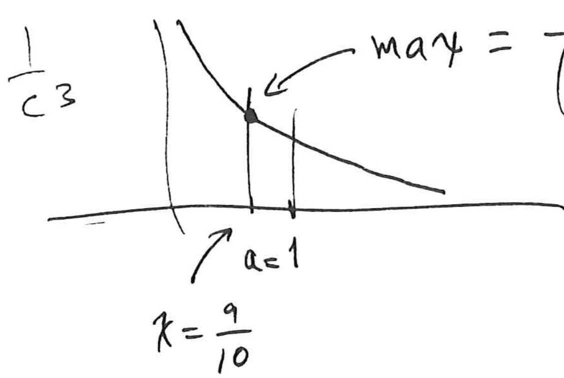
$$|R_2| = \frac{1}{3}\frac{1}{c^3}|x-1|^3$$



$$\max = \frac{1}{(9/10)^3} = \frac{10^3}{9^3}$$

3

$$|R_2| < \frac{1}{3}\frac{10^3}{9^3}|x-1|^3 = \frac{10^3}{3\cdot 9^3}\frac{1}{10^3} = \frac{1}{3\cdot 9^3} = \boxed{\frac{1}{3^7}}$$

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