Final exam / 2022.5.11 / MAT 1313.002 / Algebra and number systems

1. (a) If P, Q, R are propositions, use a truth table to prove that
   $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

   (b) If X, Y, Z are sets, prove that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

(a)

```
f(P,Q,R):=[P,Q,R,Q or R,P and (Q or R),P and Q,P and R,P and Q or P and R]$
s:[f(P,Q,R)]$
for P in [false,true]
do for Q in [false,true]
do for R in [false,true]
do (s:append(s,[f(P,Q,R)]))$
apply(matrix,s);
```

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$P \wedge Q \vee P \wedge R$
false	false	false	false	false	false	false	false
false	false	true	true	false	false	false	false
false	true	false	true	false	false	false	false
false	true	true	true	false	false	false	false
true	false	false	false	false	false	false	false
true	false	true	true	true	false	true	true
true	true	false	true	true	true	false	true
true	true	true	true	true	true	true	true

inputs

Same outputs :)

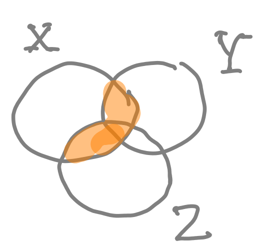
(b) $a \in X \cap (Y \cup Z) \Leftrightarrow a \in X \text{ and } a \in Y \cup Z$

$\Leftrightarrow a \in X \text{ and } (a \in Y \text{ or } a \in Z)$

$\Leftrightarrow (a \in X \text{ and } a \in Y) \text{ or } (a \in X \text{ and } a \in Z)$ (by part (a))

$\Leftrightarrow a \in X \cap Y \text{ or } a \in X \cap Z$

$\Leftrightarrow a \in (X \cap Y) \cup (X \cap Z)$:)



2. Consider the Diophantine equation $15x - 24y = 9$

(a) Find the general integer solution to the equation.

(b) Find three distinct particular integer solutions to the equation and sketch them in the plane.

```
load("gcdex")$
a:15$b:−24$c:9$[a,b,c];
[x1,y1,d]:−igcdex(a,b); /· added − so that gcd > 0 ·/
a·x1+b·y1; /· should be same as gcd ·/

/· particular solution ·/ x:x1·c/d$y:y1·c/d$[x,y];
a·x+b·y; /· should be same as c ·/

/· general solution ·/ [x+n·b/d,y−n·a/d];

puntos:create_list([x+n·b/d,y−n·a/d],n,−2,0);
load("draw")$ /· for maxima.cesga.es skip this line ·/
draw2d(proportional_axes=xy,
  grid=true,
  point_type=filled_circle,
  points(puntos))$
```

$[15,-24,9]$

$[-3,-2,3]$

3 $= -3\cdot 15 - 2(-24)$

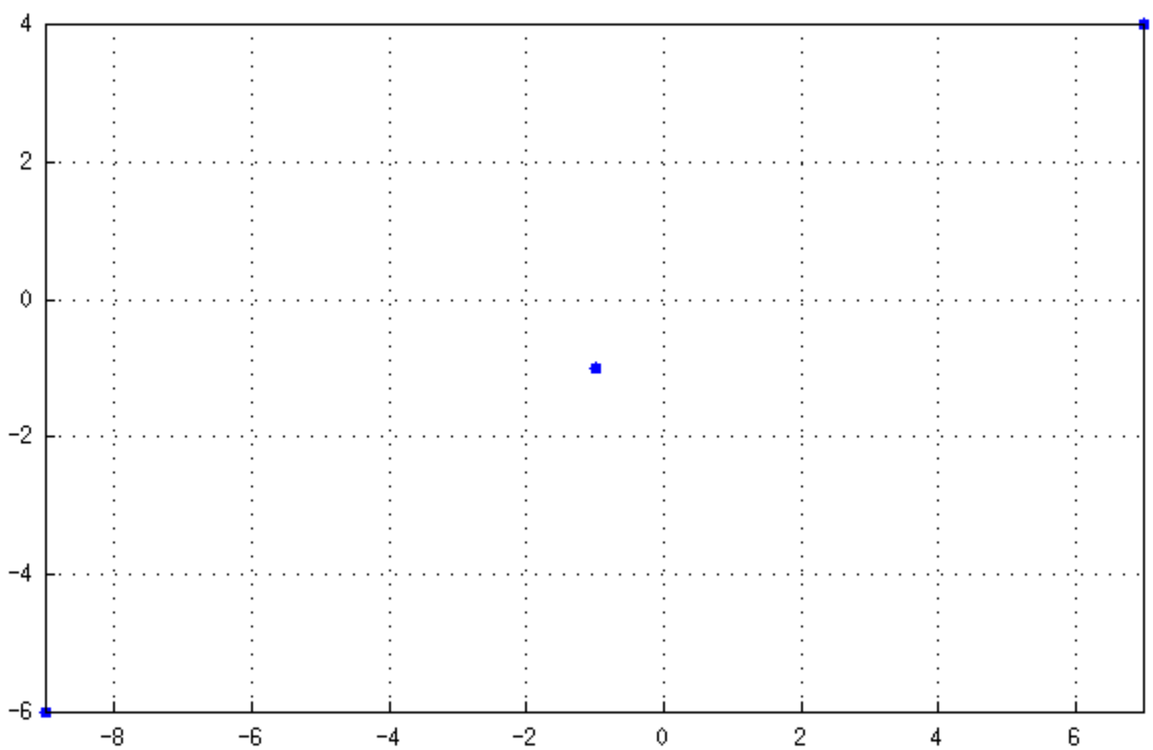
$[-9,-6]$

9

$[-8\,n-9,-5\,n-6]$

$[[7,4],[-1,-1],[-9,-6]]$

3 particular solutions to plot



3. Find all simultaneous integer solutions to the system of equations

$$2x \equiv 4 \bmod 9$$

$$3x \equiv 8 \bmod 11$$

(%i12)
```
a:[2,3];
b:[4,8];
m:[9,11];
ai:create_list(inv_mod(a[k],m[k]),k,1,2);
c:create_list(mod(ai[k]·b[k],m[k]),k,1,2);

/· Chinese remainder formula ·/
mm:prod(m[k],k,1,2);
M:mm/m;
Mi:create_list(inv_mod(M[k],m[k]),k,1,2);
apply(matrix,create_list([m[k],M[k],Mi[k],c[k],mod(M[k]·Mi[k]·c[k],mm)],k,1,2));
mod(sum(M[k]·Mi[k]·c[k],k,1,2),mm);

/· check ·/ cc:chinese(c,m);
create_list(mod(a[k]·cc,m[k]),k,1,2);
```

(a) [2,3]

(b) [4,8]

(m) [9,11]

(ai) [5,4] $\quad 2\cdot 5 \equiv 1 \bmod 9, \quad 3\cdot 4 \equiv 1 \bmod 11$

(c) [2,10] $\quad x \equiv 2 \bmod 9, \quad x \equiv 10 \bmod 11$

(mm) 99

(M) [11,9]

(Mi) [5,5] $\quad 11\cdot 5 \equiv 1 \bmod 9, \quad 9\cdot 5 \equiv 1 \bmod 11$

(%o9) $\begin{bmatrix} 9 & 11 & 5 & 2 & \underline{11} \\ 11 & 9 & 5 & 10 & \underline{54} \end{bmatrix}$ add

(%o10) 65 $\quad x \equiv 65 \bmod 99$

(cc) 65

(%o12) [4,8]

4. A sequence $a_n \in \mathbf{Z}, n \geq 0$ is defined recursively by $a_0 = 3, a_1 = 10$ and for $a > 1$

$$a_n = 7a_{n-1} - 12a_{n-2}$$

(a) Compute a_n for $n \leq 6$

(b) Prove by induction that $a_n = 2 \cdot 3^n + 4^n$ for all $n \geq 0$

```
s:[3,10]$
for k from 2 thru 6 do (s:append(s,[7·last(s)−12·s[length(s)−1]]))$s;
create_list(2·3^n+4^n,n,0,6);
```

[3,10,34,118,418,1510,5554]

$[\underbrace{3,10}_{\text{basis of induction}},34,118,418,1510,5554]$

If $n \geqslant 2$ $\quad a_n = 7a_{n-1} - 12a_{n-2}$

$$= 7(2\cdot 3^{n-1} + 4^{n-1}) - 12(2\cdot 3^{n-2} + 4^{n-2})$$

$$= 2(7\cdot 3 - 12)3^{n-2} + (7\cdot 4 - 12)4^{n-2}$$

$$= 2\cdot 9\cdot 3^{n-2} + 16\cdot 4^{n-2} = 2\cdot 3^n + 4^n$$ ☺

5. Let $p \in \mathbf{Z}[x]$, $p(x) = x^4 - x^3 + x - 1$. By inspection $p(1) = 0$. Use this to find all complex roots of p and sketch them in the complex plane.

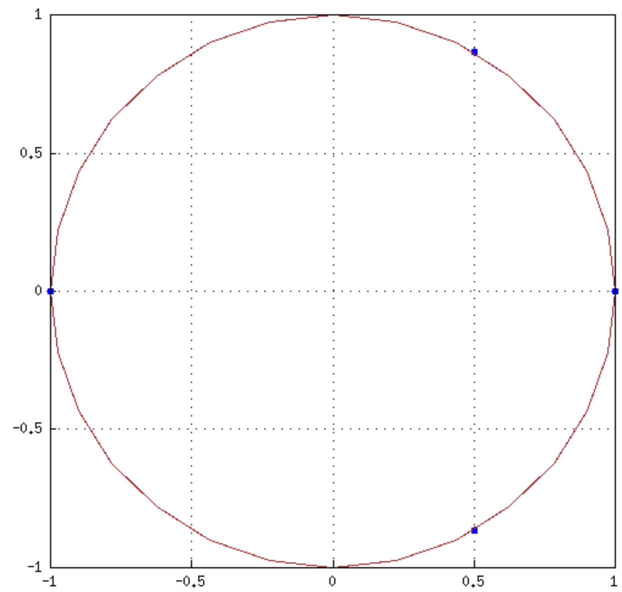
Sol. 1:

```
c2p(z):=[realpart(z),imagpart(z)]$
p2c(p):=p[1]+%i·p[2]$
csimp(z):=p2c(c2p(z))$

p(x):=x^4−x^3+x−1$p(x);
q(x):=divide(p(x),x−1)[1]$q(x);
z0:exp(%i·%pi/3);
h:create_list(exp(%i·2·%pi·k/3),k,0,2);
zz:ratsimp(z0·h);
zz:append([1],zz);
/· check ·/ ratsimp(map(p,zz));

puntos:map(c2p,zz)$
load("draw")$ /· for maxima.cesga.es skip this line ·/
circle:ellipse(0,0,1,1,0,360)$
draw2d(proportional_axes=xy,
  grid=true,
  point_type=filled_circle,
  points(puntos),
  transparent=true,color=brown,circle)$
```

Sol. 2: $p(x) = x^3(x-1) + x - 1 = (x-1)(x^3+1)$

$= (x-1)(x+1)(x^2-x+1)$

$x = 1, -1, \dfrac{1 \pm i\sqrt{3}}{2}$



$x^4 - x^3 + x - 1$

$x^3 + 1$ $\Big\}$ $x^4 - x^3 + x - 1 = (x-1)(x^3+1)$

$\dfrac{\sqrt{3}\,\%i}{2} + \dfrac{1}{2}$ particular sol. $e^{\pi i/3}$

$[1, \dfrac{\sqrt{3}\,\%i}{2} - \dfrac{1}{2}, -\dfrac{\sqrt{3}\,\%i}{2} - \dfrac{1}{2}]$ cube root of unity: $e^{2\pi i k/3}$, $k = 0, 1, 2$

$[\dfrac{\sqrt{3}\,\%i + 1}{2}, -1, -\dfrac{\sqrt{3}\,\%i - 1}{2}]$ cube root of -1: $e^{\pi i (2k+1)/3}$, $k = 0, 1, 2$

$[1, \dfrac{\sqrt{3}\,\%i + 1}{2}, -1, -\dfrac{\sqrt{3}\,\%i - 1}{2}]$ gen. sol.

$[0, 0, 0, 0]$