Midterm 2 / 2010.11.29 / MAT 1224.008 / Calculus II

Name: ______________________________

Please show all work and justify your answers.

1. Use trigonometric substitution to evaluate

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

2. Find all solutions to the following equations for y as a function of x.

$$\text{(a) } 2\sqrt{xy}\,\frac{dy}{dx} = 1, \quad x, y > 0 \qquad \text{(b) } \frac{dy}{dx} - xy = x, \quad y(0) = 3$$

3. Evaluate the following sums

$$\text{(a) } \sum_{n=1}^{\infty} \left[\frac{5}{2^n} + \frac{1}{3^n}\right] \qquad \text{(b) } \sum_{n=1}^{\infty} nx^n$$

[Hint for (b): recognize the series as x times the derivative of a known series]

4. Find Taylor series at $x = c$ and determine the interval of convergence. If you have trouble with writing out the general series, compute the first four nonzero terms for partial credit.

$$\text{(a) } \frac{x^{77}}{2-x}, \quad c = 0 \qquad \text{(b) } \ln x, \quad c = 1$$

[Hint for (a): You don't want to use Taylor's formula alone, trust me]

5. Find the first five nonzero terms of the Fourier series for the function on the interval $[-2, 2]$ defined by $f(x) = x^2$ for x between -1 and 1 and $f(x) = 0$ otherwise.

1	2	3	4	5	total (50)	%

Prelim. course grade: %

(i) $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

Let $x = 2\sin\theta$ $\qquad dx = 2\cos\theta\, d\theta$

Then $\sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{2\cancel{\cos\theta}\, d\theta}{2^2(\sin\theta)^2\, 2\cancel{\cos\theta}} = \frac{1}{4}\int (\csc\theta)^2 d\theta$$

$$= -\frac{1}{4}\cot\theta = -\frac{1}{4}\frac{\cos\theta}{\sin\theta} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

2 a) $2\sqrt{xy}\,\frac{dy}{dx} = 1$

$$\sqrt{y}\,dy = \frac{1}{2\sqrt{x}}\,dx$$

$$\frac{2}{3}y^{\frac{3}{2}} = \sqrt{x} + C$$

$$y^{\frac{3}{2}} = \frac{3}{2}(\sqrt{x} + c)$$

$$y = \left[\frac{3}{2}(\sqrt{x} + c)\right]^{2/3}$$

b) $\frac{dy}{dx} - xy = x \qquad y(0) = 3$

$p = -x \qquad q = x \qquad\qquad y' + py = q$

$$v = e^{-\frac{x^2}{2}} \qquad\qquad v = e^{\int p}$$

$$y = e^{\frac{x^2}{2}} \int e^{-\frac{x^2}{2}}\, x\, dx \qquad\qquad y = \frac{1}{v}\int vq$$

$$y = e^{\frac{x^2}{2}}\left[-e^{-\frac{x^2}{2}} + C\right]$$

$$= -1 + Ce^{\frac{x^2}{2}} \qquad\qquad y(0) = 3 \Rightarrow C = 4$$

$$\boxed{y = -1 + 4e^{\frac{x^2}{2}}}$$

Alternate technique: $y' - xy = x$

$$\frac{dy}{dx} = xy + x$$

$$\int \frac{dy}{y+1} = \int x \, dx$$

$$\ln|y+1| = \frac{x^2}{2} + C$$

$$|y+1| = e^{\frac{x^2}{2} + C}$$

$$y + 1 = \underbrace{\pm e^C}_{A} e^{\frac{x^2}{2}}$$

$$y = -1 + Ae^{x^2/2}$$

etc.

3

a) $$\sum_{n=1}^{\infty}\left[\frac{5}{2^n}+\frac{1}{3^n}\right]=\sum_{n=1}^{\infty}\frac{5}{2^n}+\sum_{n=1}^{\infty}\frac{1}{3^n}$$

$$=\frac{5/2}{1-\frac{1}{2}}+\frac{1/3}{1-\frac{1}{3}}=\frac{11}{2}$$

b) $$\sum_{n=1}^{\infty} n x^n = x\sum_{n=1}^{\infty} n x^{n-1} = x\sum_{n=1}^{\infty}\frac{d}{dx}x^n$$

$$= x\frac{d}{dx}\sum_{n=1}^{\infty}x^n = x\frac{d}{dx}\frac{x}{1-x} = x\,\frac{\cancel{1-x-x(-1)}}{(1-x)^2} = \frac{x}{(1-x)^2}$$

for $|x|<1$

(divergent otherwise)

④ a) $$\frac{x^{77}}{2-x} = \frac{x^{77}}{2}\frac{1}{1-\frac{x}{2}} = \frac{x^{77}}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{x^{n+77}}{2^{n+1}}$$

b) $\ln x$, at $x=1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$a_n = \frac{f^{(n)}(1)}{n!}$
0	$\ln x$	0	0
1	$\frac{1}{x}$	1	1
2	$-\frac{1}{x^2}$	-1	$-1/2$
3	$\frac{2}{x^3}$	2	$\frac{2}{3\cdot 2} = \frac{1}{3}$
4	$-\frac{3\cdot 2}{x^4}$	$-3\cdot 2$	$-\frac{3\cdot 2}{4\cdot 3\cdot 2} = -\frac{1}{4}$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \dots$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$$

Ratio: $$\left|\frac{\frac{(-1)^{n+2}}{n+1}(x-1)^{n+1}}{\frac{(-1)^{n+1}}{n}(x-1)^n}\right| = \frac{n}{n+1}|x-1| \to |x-1|$$

Radius = 1

Endpoints: $x=0$ $$\sum\frac{(-1)^{n+1}(-1)^n}{n} = \sum\frac{(-1)^{2n+1}}{n} = -\sum\frac{1}{n}$$ div

$x=2$ $\sum\frac{(-1)^{n+1}}{n}$ conv. by the alternating series test

Interval: $0 < x \leq 2$

Alternate technique:

$\ln x$ at $x=1$

$$\ln x = \int \frac{1}{x}\,dx = \int \frac{dx}{1+x-1} = \int \frac{dx}{1-(-(x-1))}$$

conv for $|x-1|<1$

$$= \int \sum_{n=0}^{\infty} (-(x-1))^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int (x-1)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} + C$$

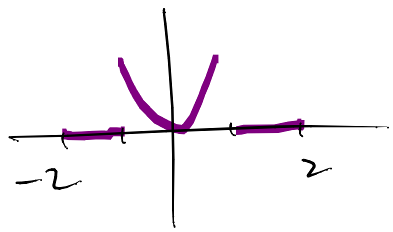
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} + C$$

To find C, plug in $x=1$ $\qquad 0 = 0 + C \quad \therefore C = 0$

$$\therefore \ln x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

(5) $f(x) = x^2$ for x between -1 and 1

0 otherwise on $[-2,2]$



Since f is even, all b_n's $= 0$

$$a_0 = \frac{1}{4}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\int_0^1 x^2\,dx = \frac{1}{6}$$

$$a_n = \frac{1}{2}\int_{-1}^{1} \underbrace{x^2\cos\left(\frac{n\pi x}{2}\right)}_{\text{even}}\,dx = \int_0^1 x^2\cos\left(\frac{n\pi x}{2}\right)dx$$

x^2		$\cos\left(\frac{n\pi x}{2}\right)$
$2x$	$+$	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
2	$-$	$-\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$
0	$+$	$-\frac{8}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)$

$$= \left[\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{8x}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{16}{n^3\pi^3}\sin\left(\frac{n\pi x}{2}\right)\right]_0^1$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{16}{n^3\pi^3}\sin\left(\frac{n\pi}{2}\right)$$

$$= \left(\frac{2}{n\pi} - \frac{16}{n^3 \pi^3} \right) \sin\left(\frac{n\pi}{2}\right) + \frac{8}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right)$$

$$a_1 = \frac{2}{\pi} - \frac{16}{\pi^3} \qquad a_2 = -\frac{2}{\pi^2}$$

$$a_3 = -\frac{2}{3\pi} + \frac{16}{27\pi^3} \qquad a_4 = \frac{1}{2\pi^2}$$

$$f(x) = a_0 + a_1 \cos\left(\frac{\pi x}{2}\right) + a_2 \cos(\pi x) + a_3 \cos\left(\frac{3\pi x}{2}\right)$$

$$+ a_4 \cos(2\pi x) + \dots$$

where a_n's are as above.

Calculus II Spring 2010 Midterm 2 #1

```
> 1/x^2/sqrt(4-x^2); int(%,x): simplify(%);
```

$$\frac{1}{x^2 \sqrt{4-x^2}}$$

$$-\frac{\sqrt{4-x^2}}{4\,x}$$

#2

```
> 2*sqrt(x*y(x))*D(y)(x)=1; dsolve(%,y(x)); solve(%,y(x))[1];
```

$$2\sqrt{x\,y(x)}\;D(y)(x)=1$$

$$\frac{2\,(x\,y(x))^{(3/2)}}{x^{(3/2)}}-3\sqrt{x}-\_C1=0$$

$$\frac{(12\,x^2+4\,\_C1\,x^{(3/2)})^{(2/3)}}{4\,x}$$

```
> {D(y)(x)-x*y(x)=x,y(0)=3}; dsolve(%,y(x));
```

$$\{y(0)=3,\;D(y)(x)-x\,y(x)=x\}$$

$$y(x)=-1+4\,e^{\left(\frac{x^2}{2}\right)}$$

#3

```
> 5/2^n+1/3^n; sum(%,n=1..infinity);
```

$$\frac{5}{2^n}+\frac{1}{3^n}$$

$$\frac{11}{2}$$

```
> n*x^n; sum(%,n=1..infinity);
```

$$n\,x^n$$

$$\frac{x}{(x-1)^2}$$

#4

```
> Order:=83: x^77/(2-x); series(%,x=0); Order:=6:
```

$$\frac{x^{77}}{2-x}$$

$$\frac{1}{2}x^{77}+\frac{1}{4}x^{78}+\frac{1}{8}x^{79}+\frac{1}{16}x^{80}+\frac{1}{32}x^{81}+\frac{1}{64}x^{82}+O(x^{83})$$

```
> ln(x); series(%,x=1);
```

$$\ln(x)$$

$$x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + O((x-1)^6)$$

#5

```
> a0:=int(x^2,x=-1..1)/4;
```

$$a0 := \frac{1}{6}$$

```
> an:=(1/2)*int(x^2*cos(n*Pi*x/2),x=-1..1);
```

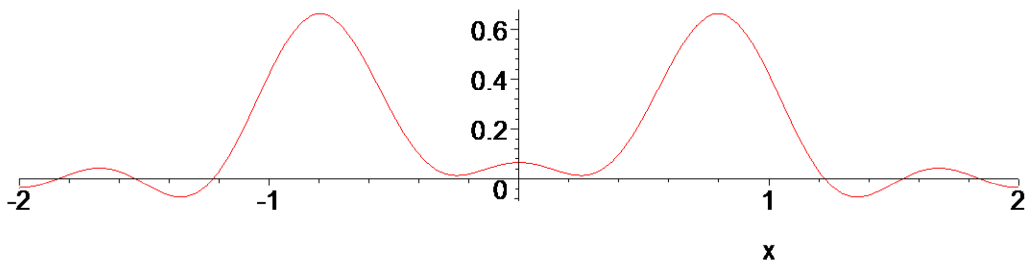
$$an := \frac{2\left(n^2\pi^2\sin\left(\frac{n\pi}{2}\right) - 8\sin\left(\frac{n\pi}{2}\right) + 4n\pi\cos\left(\frac{n\pi}{2}\right)\right)}{n^3\pi^3}$$

```
> a:=[seq((1/2)*int(x^2*cos(n*Pi*x/2),x=-1..1),n=1..5)];
```

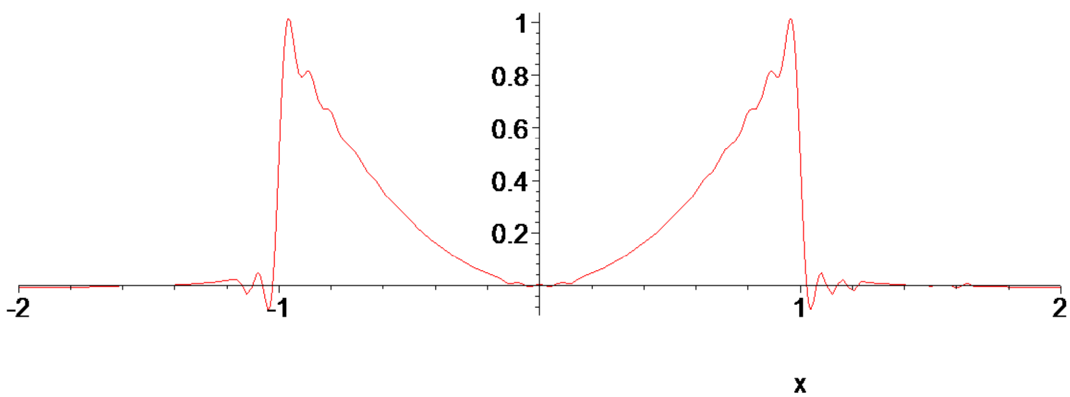
$$a := \left[\frac{2(\pi^2-8)}{\pi^3}, -\frac{2}{\pi^2}, -\frac{2(9\pi^2-8)}{27\pi^3}, \frac{1}{2\pi^2}, \frac{2(25\pi^2-8)}{125\pi^3}\right]$$

```
> a0+sum(a[n]*cos(n*Pi*x/2),n=1..5);
  plot(%,x=-2..2,scaling=constrained);
```

$$\frac{1}{6} + \frac{2(\pi^2-8)\cos\left(\frac{\pi x}{2}\right)}{\pi^3} - \frac{2\cos(\pi x)}{\pi^2} - \frac{2}{27}\frac{(9\pi^2-8)\cos\left(\frac{3\pi x}{2}\right)}{\pi^3} + \frac{1}{2}\frac{\cos(2\pi x)}{\pi^2}$$
$$+ \frac{2}{125}\frac{(25\pi^2-8)\cos\left(\frac{5\pi x}{2}\right)}{\pi^3}$$




```
> a:=[seq((1/2)*int(x^2*cos(n*Pi*x/2),x=-1..1),n=1..50)]:
  a0+sum(a[n]*cos(n*Pi*x/2),n=1..50):
  plot(%,x=-2..2,scaling=constrained);
```


1
0.8
0.6
0.4
0.2
-2
-1
1
2
x

>

>