1. The force between two atoms as a function of the distance $x > 0$ between the atoms is given by $-a/x^2 + b/x^3$, where a and b are positive constants. Which value of x minimizes the force?

Let $f = -\frac{a}{x^2} + \frac{b}{x^3} = -ax^{-2} + bx^{-3}$. Then $f' = 2ax^{-3} - 3bx^{-4}$

$= x^{-4}(2ax - 3b)$. $f' = 0 \Rightarrow$ $\boxed{x = \frac{3b}{2a}}$

2. Find indefinite integrals of the following functions

(a) $\frac{e^{-x}}{2+e^{-x}}$ (b) $\frac{t}{e^t} = te^{-t}$

a) Let $u = 2 + e^{-x}$. Then $du = -e^{-x}dx$, so

$$\int \frac{e^{-x}}{2+e^{-x}}dx = -\int \frac{1}{u}du = -\ln u = \boxed{-\ln(2+e^{-x}) + C}$$

b)

t	$\oplus$	e^{-t}
1	$\ominus$	$-e^{-t}$
0		e^{-t}

$$\int \frac{t}{e^t}dt = -te^{-t} - e^{-t}$$

$$= \boxed{-\frac{t}{e^t} - \frac{1}{e^t} + C}$$

3. Determine whether the improper integral $\int_0^1 \frac{1}{\sqrt{x}+x^2}dx$ converges or diverges. Justify your assertion by comparison to an integral whose convergence or divergence can be determined directly.

$\sqrt{x} + x^2 \geq \sqrt{x}$ so $\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{\sqrt{x}}$ So $\int_0^1 \frac{1}{\sqrt{x}+x^2}dx$

$$\leq \int_0^1 \frac{1}{\sqrt{x}}dx = \int_0^1 x^{-1/2}dx = 2x^{\frac{1}{2}}\Big|_0^1 = 2$$

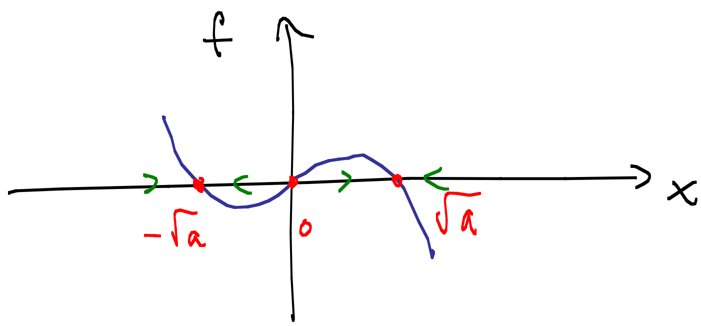
4. For the autonomous differential equation $dx/dt = ax - x^3$, where a is a positive constant, draw the phase-line diagram, find the equilibria, and determine their stability.

$$f = ax - x^3 = x(a - x^2) = x(\sqrt{a} - x)(\sqrt{a} + x)$$

$$\therefore \frac{dx}{dt} = 0 \Rightarrow x = 0, \sqrt{a}, \text{ or } -\sqrt{a} \quad \leftarrow \text{equilibria}$$

$$f' = a - 2x^2, \quad f'(0) = a > 0 \quad \therefore x = 0 \text{ is unstable}$$

$$f'(\pm\sqrt{a}) = a - 2a = -a < 0 \quad \therefore x = \pm\sqrt{a} \text{ are stable}$$



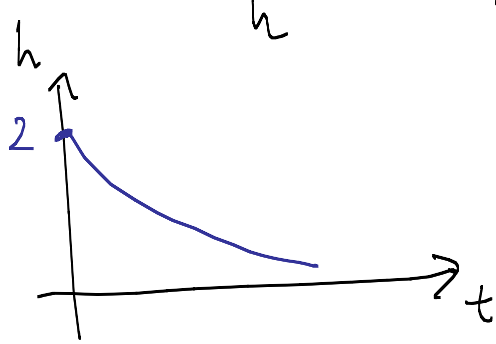
5. Solve the differential equation $dh/dt = -h^2$ with initial condition $h(0) = 2$. Sketch the solution and describe its long-term behavior.

$$\int \frac{dh}{h^2} = -\int dt = -t + c$$

$$\int h^{-2}\, dh = -h^{-1} \qquad \therefore -\frac{1}{h} = -t + c$$

$$h(0) = 2 \Rightarrow -\frac{1}{2} = c$$

$$\therefore -\frac{1}{h} = -t - \frac{1}{2} \qquad \text{so} \quad \boxed{h = \frac{1}{t + \frac{1}{2}}}$$



In the long run h decreases towards 0.

MAT 1194.001 Fall 2009 Midterm 2

#1

```
> f:=-a/x^2+b/x^3;
```

$$f := -\frac{a}{x^2} + \frac{b}{x^3}$$

```
> diff(f,x); xmin:=solve(%,x);
```

$$\frac{2\,a}{x^3} - \frac{3\,b}{x^4}$$

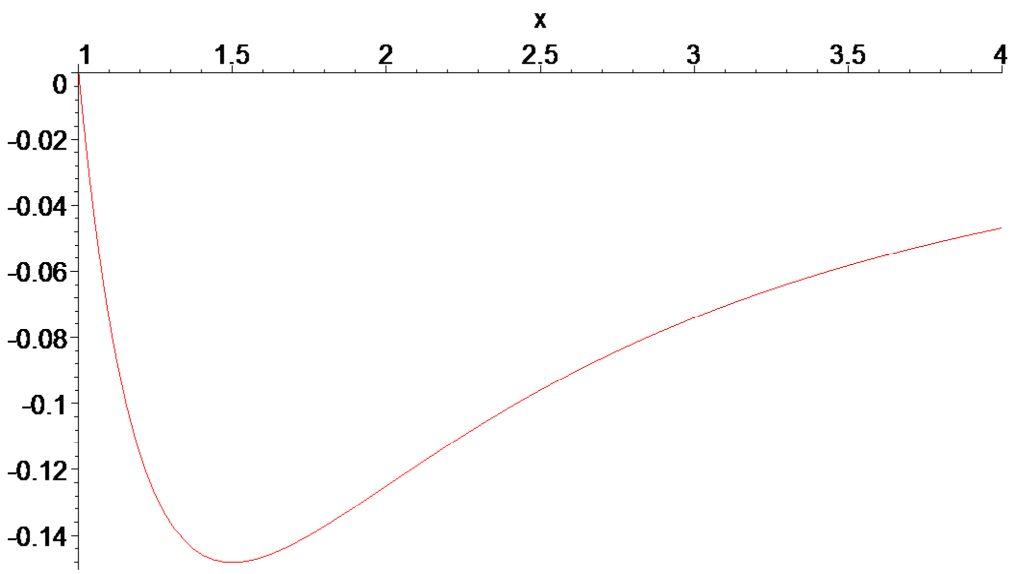
$$xmin := \frac{3\,b}{2\,a}$$

```
> diff(diff(f,x),x); subs(x=xmin,%);
```

$$-\frac{6\,a}{x^4} + \frac{12\,b}{x^5}$$

$$\frac{32\,a^5}{81\,b^4}$$

```
> plot(subs({a=1,b=1},f),x=1..4);
```



#2

```
> exp(-x)/(2+exp(-x)); int(%,x);
```

$$\frac{\mathbf{e}^{(-x)}}{2 + \mathbf{e}^{(-x)}}$$

$$-\ln(2 + \mathbf{e}^{(-x)})$$

```
> t/exp(t); int(%,t);
```

$$\frac{t}{\mathbf{e}^t}$$

$$-\frac{t}{e^t}-\frac{1}{e^t}$$

#3

```
> 1/(sqrt(x)+x^2); int(%,x=0..1);
```

$$\frac{1}{\sqrt{x}+x^2}$$

$$\frac{2\sqrt{3}\,\pi}{9}+\frac{2}{3}\ln(2)$$

#4

```
> f:=a*x-x^3;
```

$$f := a\,x-x^3$$

```
> equi:=solve(f,x);
```

$$equi := 0, \sqrt{a}, -\sqrt{a}$$

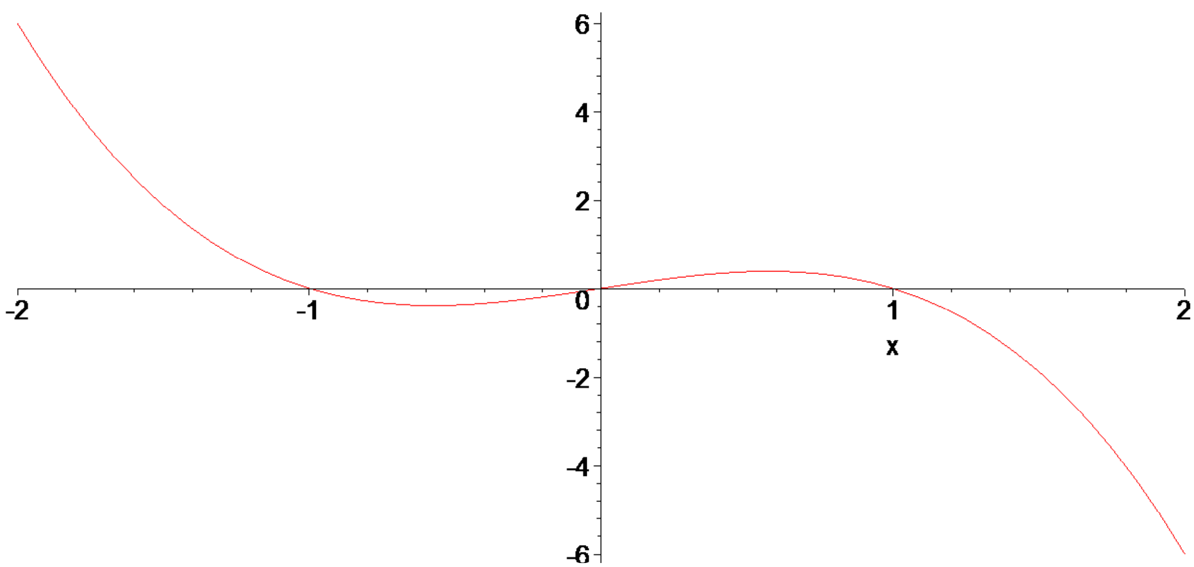
```
> df:=diff(f,x);
```

$$df := a-3\,x^2$$

```
> seq(subs(x=equi[i],df),i=1..3);
```

$$a, -2\,a, -2\,a$$

```
> plot(subs(a=1,f),x=-2..2);
```



#5

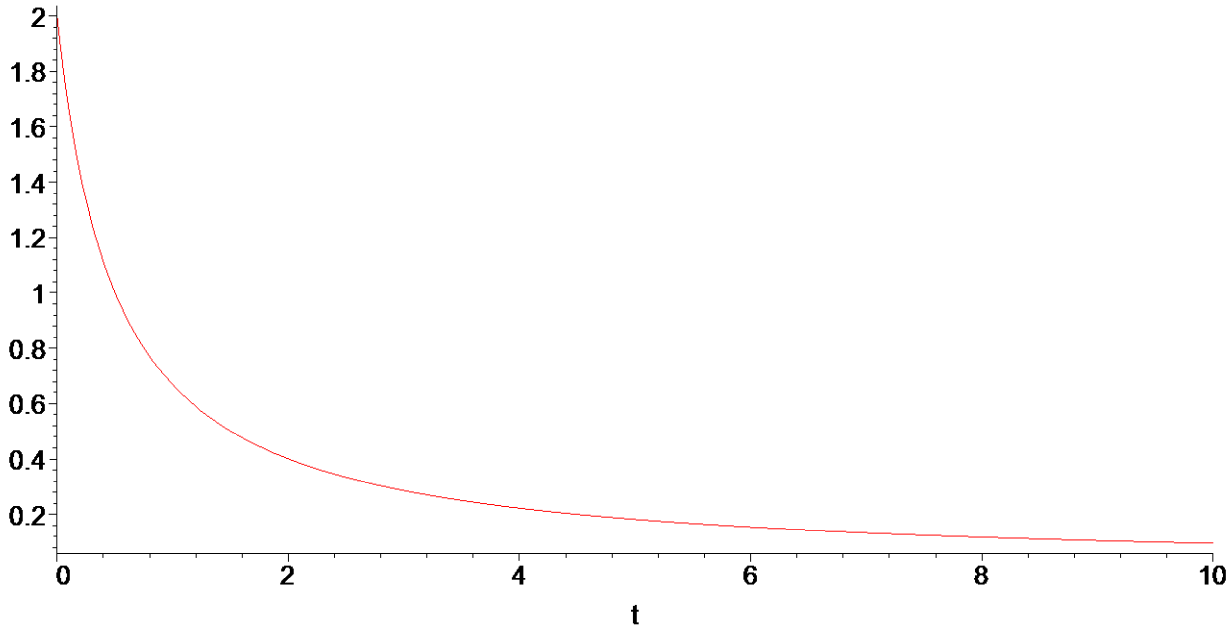
```
> diff(h(t),t)=-h(t)^2; dsolve({%,h(0)=2},h(t)); simplify(%);
  hh:=subs(%,h(t)):
```

$$\frac{d}{dt}\,\mathrm{h}(t)=-\mathrm{h}(t)^2$$

$$\mathrm{h}(t)=\frac{1}{t+\frac{1}{2}}$$

$$h(t) = \frac{2}{2t+1}$$

```
> plot(hh,t=0..10);
```




```
>
```