Final exam / 2009.12.11 / Calculus for the Biosciences / MAT 1194.001

Note Title 12/15/2009

1. A radioactive isotope has a half-life of 8 days. If the initial amount is 5 grams, how long will it take for the amount to decrease to 2 grams?

$b(t) = 5 \cdot 2^{-t/8}$

If $b(t) = 2$,

then $5 \cdot 2^{-t/8} = 2$

so $\ln 5 - \frac{t}{8} \ln 2 = \ln 2$

so $t = \frac{8(\ln 5 - \ln 2)}{\ln 2} \approx \boxed{10.575}$

More detail:

$b(t) = Ae^{kt}$, $b(0) = 5 = A$

Also $b(8) = \frac{A}{2} = Ae^{8k}$

So $8k = \ln\frac{1}{2} = -\ln 2$, so $k = -\frac{\ln 2}{8}$

So $b(t) = 5e^{-\frac{\ln 2}{8}t}$

$= 5e^{\ln 2^{-t/8}} = 5 \cdot 2^{-t/8}$

Check in Maple:

```
> b:=5*2^(-t/8);
```

$b := 5\,2^{\left(-\frac{t}{8}\right)}$

Check:

```
> subs(t=8,b);
```

$\frac{5}{2}$

```
> solve(b=2,t); evalf(%);
```

$-\frac{8\ln\left(\frac{2}{5}\right)}{\ln(2)}$

10.57542476

#2

```
> cos(1+exp(2*x)); diff(%,x);
```

$\cos(1 + e^{(2x)})$

$-2\sin(1 + e^{(2x)})\,e^{(2x)}$

```
> ln(x)/(2*x+1); diff(%,x); simplify(%);
```

$\frac{\ln(x)}{2x+1}$

$\frac{1}{x(2x+1)} - \frac{2\ln(x)}{(2x+1)^2}$

$-\frac{-2x - 1 + 2\ln(x)x}{x(2x+1)^2}$

2. Find the derivatives of

(a) $\cos(1 + e^{2x})$ (b) $\frac{\ln x}{2x+1}$

a) $-\sin(1 + e^{2x}) \cdot e^{2x} \cdot 2$

b) $\frac{\frac{1}{x}(2x+1) - \ln x \cdot 2}{(2x+1)^2}$

3. For the Ricker model for fish population $x_{t+1} = rx_te^{-2x_t}$ find the equilibria. For which values of r is each equilibrium stable? Unstable?

Solve $x = \underbrace{rxe^{-2x}}_{f(x)}$ for x: $\boxed{x=0}$ or $1 = re^{-2x}$, so

$-2x = \ln\left(\frac{1}{r}\right)$, so $\boxed{x = \frac{\ln r}{2}}$

$f' = r\left(e^{-2x} + xe^{-2x}(-2)\right) = re^{-2x}(1-2x)$

$f'(0) = r$, so $x=0$ is stable when $r<1$ and unstable for $r>1$.

$f'\left(\frac{\ln r}{2}\right) = r \cdot \frac{1}{r}(1-\ln r) = 1 - \ln r$, so $\frac{\ln r}{2}$ is stable $|1-\ln r| < 1$, i.e. $-1 < 1-\ln r < 1$

$0 < \ln r < 2$

$1 < r < e^2 \approx 7.4$

(unstable when $r<1$ or $r > e^2$)

```
> f:=r*x*exp(-2*x);
```

$$f := r\,x\,e^{(-2\,x)}$$

```
> equi:=[solve(x=f,x)];
```

$$equi := \left[0, -\frac{1}{2}\ln\left(\frac{1}{r}\right)\right]$$

```
> diff(f,x); df:=simplify(%);
```

$$r\,e^{(-2\,x)} - 2\,r\,x\,e^{(-2\,x)}$$

$$df := -r\,e^{(-2\,x)}\,(-1+2\,x)$$

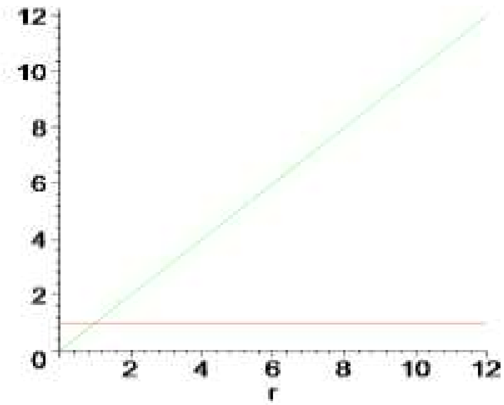
Plug in equilibria into the derivative of the updating function, plot the absolute values as functions of r:

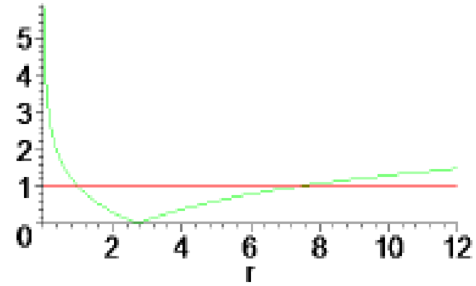
```
> for i from 1 to 2 do subs(x=equi[i],df); print(simplify(%)); od:
```

$$r$$

$$1 + \ln\left(\frac{1}{r}\right)$$

```
> for i from 1 to 2 do subs(x=equi[i],df);
  print(plot({abs(%),1},r=0..12,scaling=constrained)); od:
```





Find the points of intersection (and their floats):

```
> for i from 1 to 2 do subs(x=equi[i],df); s:=solve(abs(%)=1,r);
  print(s); print(evalf(s)); od:
```

$$1, -1$$

$$1., -1.$$

$$1, \frac{1}{e^{(-2)}}$$

$$1., 7.389056101$$

4. Let $f(t) = t - t^3$. Find all the critical points of f on the interval $0 \le t \le 2$. Use the second derivative to determine concavity at the critical points. Find the global minimum and the global maximum of f on the interval. Where do they occur?

$f'(t) = 1 - 3t^2 \qquad f'(t) = 0 \Rightarrow t = \pm \frac{1}{\sqrt{3}} \approx \pm 0.577$

$\therefore$ $t = \frac{1}{\sqrt{3}}$ is the only critical point in the interval

$f''(t) = -6t \quad f''\left(\frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} < 0 \quad \therefore$ local max

	t	$f(t)$	
endpts	0	0	
	2	-6	$\Leftarrow$ min
critical pt.	$\frac{1}{\sqrt{3}}$	$\frac{2}{3\sqrt{3}} \approx 0.4$	$\Leftarrow$ max

```
> f:=t-t^3;
```

$$f := t - t^3$$

```
> df:=diff(f,t); [solve(%,t)]; crit:=%[2]; evalf(%);
```

$$df := 1 - 3t^2$$

$$\left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$$

$$crit := \frac{\sqrt{3}}{3}$$

$$0.5773502693$$

Concavity at the critical point

```
> diff(df,t); subs(t=crit,%);
```

$$-6\,t$$

$$-2\sqrt{3}$$

Values of f(t) at the endpoints and the critical point

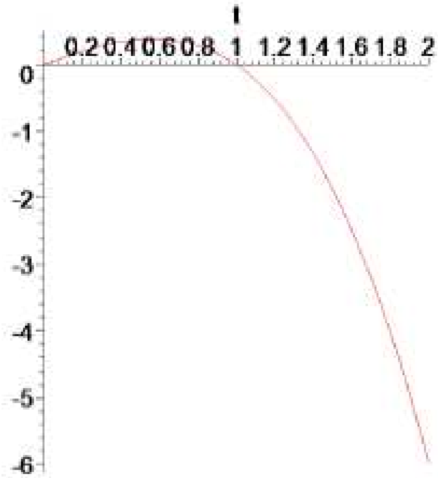
```
> {crit} union {0} union {2}: convert(%,list);
  map(xx->subs(t=xx,f),%); evalf(%);
```

$$\left[0,\,2,\,\frac{\sqrt{3}}{3}\right]$$

$$\left[0,\,-6,\,\frac{2\sqrt{3}}{9}\right]$$

$$[0.,\,-6.,\,0.3849001795]$$

```
> plot(f,t=0..2);
```



5. Find indefinite integrals of the following functions

$$\text{(a) } \frac{1}{x\ln x} \qquad \text{(b) } t^2\sin(3t)$$

a) Let $u = \ln x$, then $du = \frac{1}{x}\,dx$

we get $\int \frac{1}{u}\,du = \ln u = \ln(\ln x) + C$

b)

$$\begin{array}{cc} t^2 & \sin(3t) \\ 2t & -\frac{1}{3}\cos(3t) \\ 2 & -\frac{1}{9}\sin(3t) \\ 0 & \frac{1}{27}\cos(3t) \end{array}$$

$\frac{d}{dt}$ $\quad$ $\int dt$ $\quad$ $+$ $\quad$ $-$ $\quad$ $+$

We get $-\frac{t^2}{3}\cos(3t) + \frac{2}{9}t\sin(3t) + \frac{2}{27}\cos(3t) + C$

```
> 1/(x*ln(x)); int(%,x);
```

$$\frac{1}{x\ln(x)}$$

$$\ln(\ln(x))$$

```
> t^2*sin(3*t); int(%,t);
```

$$t^2\sin(3\,t)$$

$$-\frac{1}{3}t^2\cos(3\,t) + \frac{2}{27}\cos(3\,t) + \frac{2}{9}t\sin(3\,t)$$

6. Determine whether the improper integral $\int_0^1 \frac{1}{\sqrt[3]{x}+\sqrt{x}}\,dx$ converges by comparing it to an integral which can be computed explicitly.

$$\int_0^1 \frac{1}{\sqrt[3]{x}+\sqrt{x}}\,dx \le \int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \int_0^1 x^{-\frac{1}{3}}\,dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_0^1 = \frac{3}{2}$$

$\therefore$ converges

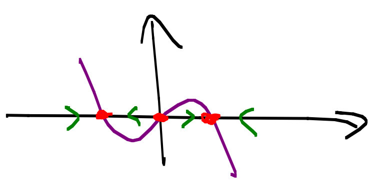
```
> f:=1/(x^(1/3)+x^(1/2)); int(f,x=0..1); evalf(%);
```

$$f := \frac{1}{x^{(1/3)}+\sqrt{x}}$$

$$5 - 6\ln(2)$$

$$0.841116916$$

7. For the autonomous differential equation $dx/dt = x - ax^3$, where a is a positive constant, draw the phase-line diagram, find the equilibria, and determine their stability, both from the diagram and by using the stability theorem.



$$\frac{dx}{dt} = 0 \Rightarrow \underbrace{x - ax^3}_{f} = x(1-ax^2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \pm\frac{1}{\sqrt{a}}$$

$f' = 1 - 3ax^2$ $\quad f'(0) = 1 > 0$ $\therefore$ $x = 0$ is unstable

$f'\left(\pm\frac{1}{\sqrt{a}}\right) = -2 < 0$ $\therefore$ both $\frac{1}{\sqrt{a}}$ and $-\frac{1}{\sqrt{a}}$ are stable.

```
> f:=x-a*x^3;
```

$$f := x - a\,x^3$$

```
> equi:=[solve(f=0,x)];
```

$$equi := \left[0, \frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}}\right]$$

```
> df:=diff(f,x);
```

$$df := 1 - 3\,a\,x^2$$

```
> map(xx->subs(x=xx,df),equi);
```

[1, -2, -2]

8. Solve the differential equation $dh/dt = -h^2$ with initial condition $h(0) = 3$. Sketch a graph of the solution $h(t)$ for $t \geq 0$. What is the limit of $h(t)$ as $t \to \infty$?

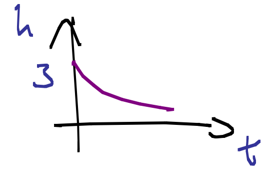
$$\int \frac{dh}{h^2} = -\int dt = -t + c$$

$$" \int h^{-2}\,dh = -h^{-1}$$

$$\therefore \ h^{-1} = t - c$$

$$h(0) = 3 \Rightarrow -c = \frac{1}{3}$$

$$\therefore \ h = \frac{1}{t + \frac{1}{3}}$$



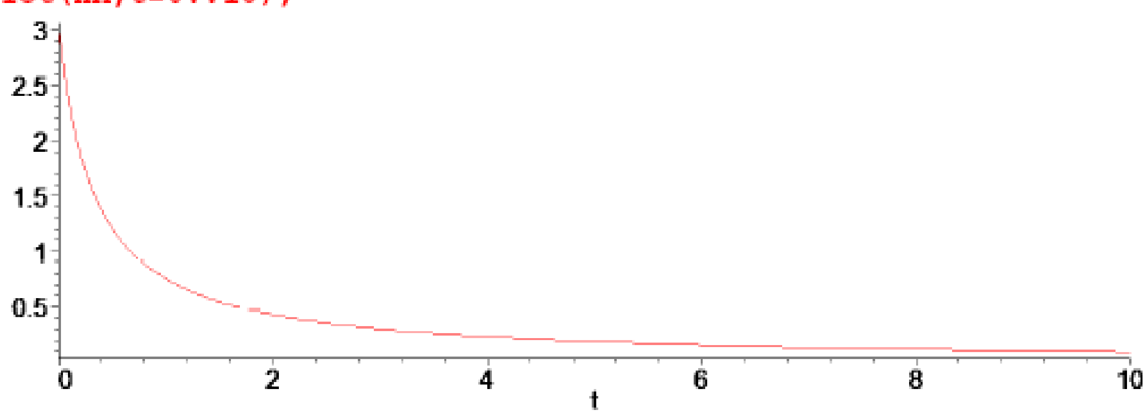
$$\lim_{t \to \infty} h(t) = 0$$

```
> diff(h(t),t)=-h(t)^2;
  dsolve({%,h(0)=3},h(t));
  hh:=subs(%,h(t)):
```

$$\frac{d}{dt}\mathrm{h}(t) = -\mathrm{h}(t)^2$$

$$\mathrm{h}(t) = \frac{1}{t + \frac{1}{3}}$$

```
> plot(hh,t=0..10);
```




```
> limit(hh,t=infinity);
```

$$0$$

Enjoy your break! — d