CS 3333.001 Midterm 2 Fall 2014

① $n > 0$ $$\sum_{k=1}^{n} k^2 = \frac{1}{6} n (n+1)(2n+1)$$

Basis: $n = 1$: $$1 \stackrel{?}{=} \frac{1}{6} \cdot 2 \cdot 3 = 1$$

Let $n > 1$. Assume true for $n-1$, i.e.

$$\sum_{k=1}^{n-1} k^2 = \frac{1}{6}(n-1)\left[(n-1)+1\right]\left[2(n-1)+1\right]$$
$$= \frac{1}{6}(n-1)n(2n-1)$$

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{1}{6}(n-1)n(2n-1) + n^2$$

$$= \frac{1}{6} n \left[(n-1)(2n-1) + 6n\right]$$

$$= \frac{1}{6} n \left[2n^2 - 2n - n + 1 + 6n\right]$$

$$= \frac{1}{6} n \left(2n^2 + 3n + 1\right)$$

$$\frac{1}{6} n (n+1)(2n+1) = \frac{1}{6} n \left(2n^2 + 2n + n + 1\right)$$

② $2x + 3y + 5 = 0 \qquad x + y + z = 5$

a) $A = \left[\begin{array}{ccc|c} 2 & 3 & 0 & -5 \\ 1 & 1 & 1 & 5 \end{array}\right]$ Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 0 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -15 \end{array}\right]$$

$$\text{rref}(A) = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 20 \\ 0 & 1 & -2 & -15 \end{array}\right]$$

z = free var.

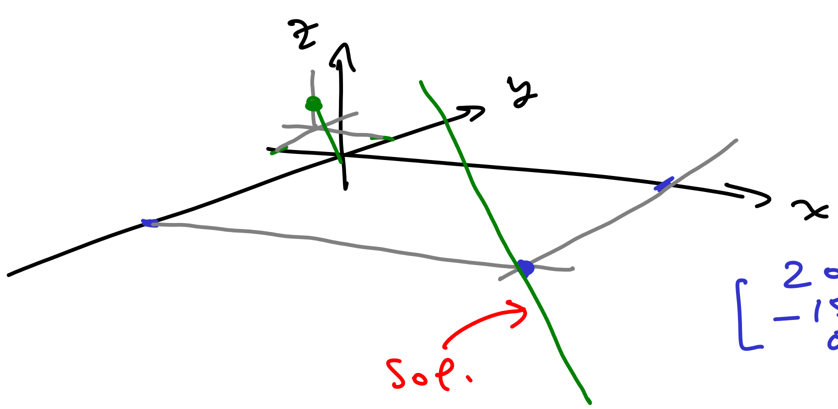
$x + 3z = 20$
$y - 2z = -15$

b) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\begin{bmatrix} 20 - 3z \\ 2z - 15 \\ z \end{bmatrix}} = \begin{bmatrix} 20 \\ -15 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

c) $$2x + 3y + 5 = 2(20 - 3z) + 3(2z - 15) + 5$$
$$= 40 - \cancel{6z} + \cancel{6z} - 45 + 5 = 0$$

$$x + y + z = 20 - \cancel{3z} + \cancel{2z} - 15 + \cancel{z} = 5$$

d)



$\begin{bmatrix} 20 \\ -15 \\ 0 \end{bmatrix}$

③ My picks:

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(see #4)

$\det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

no real roots

④ $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

a) $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix}$

$= (2-\lambda)(-1-\lambda) - 4 = -2 - 2\lambda + \lambda + \lambda^2 - 4$

$= \boxed{\lambda^2 - \lambda - 6}$ $\boxed{\lambda = 3, -2}$

b) $\underline{\lambda = 3}$, $A - 3I = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}$

rref $= \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ $x - 4y = 0$

y is free

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ y \end{bmatrix} = y \underline{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}$

$\underline{\lambda = -2}$ $A + 2I = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}$

rref $= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

c) Let $P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \quad \downarrow \text{ Gauss-Jordan}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 4 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -5 & 1 & -4 \end{array}\right]$$

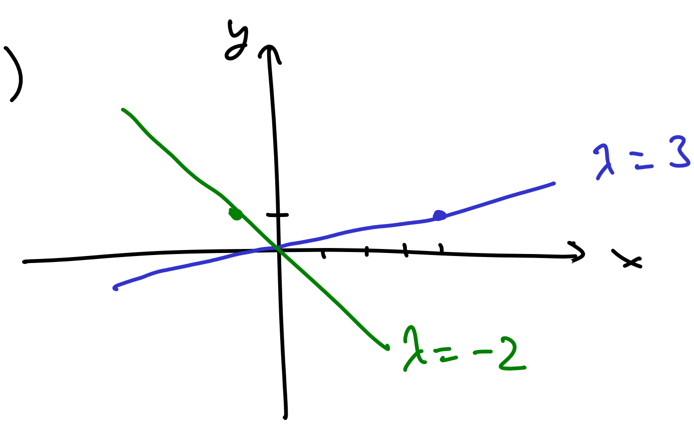
$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{4}{5} \end{array}\right]$$

$$\text{rref} = \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & -\frac{1}{5} & \frac{4}{5} \end{array}\right]$$

$$\therefore P^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1/5 & 1/5 \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 12 & 2 \\ 3 & -2 \end{bmatrix}} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

d)



Dilation by 3 along the blue line

Dilation by 2 and a flip along the green line.