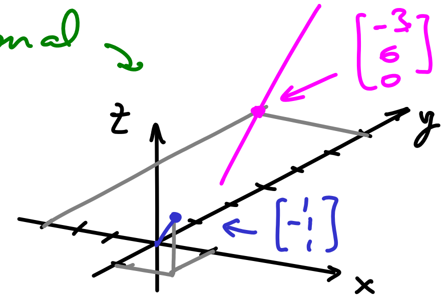
Midterm 2 — CS3333 Sp. '14

1. a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 6 \end{bmatrix} \to \boxed{\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 6 \end{bmatrix}}$

b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{\begin{bmatrix} z-3 \\ -z+6 \\ z \end{bmatrix}} = \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ← optional



Check: $x+y = z-3-z+6 = 3$ ✓

$2x+y = 2z-6-z+6 = z$ ✓

2. a-b) $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{bmatrix} \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \to \cdots$

upper triangular ⇒ $\det = \boxed{2}$; $x \mapsto Ax$ doubles volumes

$\to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$ ← A^{-1}

c) $AX = B \Rightarrow X = A^{-1}B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}}$

3. a) $\det(A-\lambda I) = \det\begin{bmatrix} -2-\lambda & 3 \\ 6 & 1-\lambda \end{bmatrix} = (\lambda-1)(\lambda+2) - 18 = \lambda^2+\lambda-20$, $\lambda = \boxed{-5, 4}$

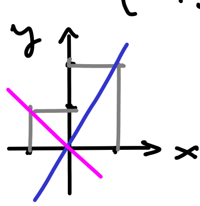
b) If $\lambda = -5$, $A - \lambda I = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

If $\lambda = 4$, $A - \lambda I = \begin{bmatrix} -6 & 3 \\ 6 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y \\ y \end{bmatrix} = \frac{1}{2} y \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

c) $\left[\begin{array}{c|c} \underbrace{\boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}}}_{P} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right] \to \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \to \left[\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \underbrace{\begin{matrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{matrix}}_{P^{-1}} \end{array}\right]$

check: $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}$

d)



$x \mapsto Ax$: dilation by 4 along the blue line, dilation by 5 and reflection along the magenta line.