Final CS 3333 Sp. '14

(1) $69 = 64 + 5 = 2^6 + 5 = 8^2 + 5 = 4\cdot 16 + 5$

$5 = 4 + 1 = 2^2 + 1$ $\therefore$ $69 =$ binary $\underline{1000101}$ = octal $\underline{105}$ = x$\underline{45}$

x$ACA = 10\cdot 16^2 + 12\cdot 16 + 10 = \underline{2762}$

(2) $77 = 76 + 1$, $1 = 77 - 76$, $\therefore$ $\gcd(77,76) = \underline{1}$, Bézout: $\underline{1}, \underline{-1}$

(3) $x \equiv 2 \bmod 4$, $2x \equiv 3 \bmod 13 \iff x \equiv 8 \bmod 13$, $x \equiv 1 \bmod 5$

$m = 4\cdot 13\cdot 5 = 260$.

i	m_i	m/m_i	mod m_i	inv.	r.h.s.	m_i inv rhs
1	4	65	1	1	2	130
2	13	20	7	2	8	320
3	5	52	2	3	1	156

sum $= 606 \equiv \boxed{86 \bmod 260}$

(4)

n	n^n	$(n+1)!$
1	1	2
2	4	6
3	27	24
4	256	120

Conjecture $n \geq 3 \Rightarrow n^n > (n+1)!$

Basis of induction: $n=3$: $27 > 24$ ☺

Inductive step: Let $n > 3$ and assume $(n-1)^{n-1} > n!$

$n^n = n^{n-1} n = (n-1+1)^{n-1} n =$ ← Binomial theorem

$= [(n-1)^{n-1} + (n-1)(n-1)^{n-2} + \dots + 1]n > [(n-1)^{n-1} + (n-1)(n-1)^{n-2}]n =$

$= 2(n-1)^{n-1} n > 2n!\,n = (n+n)n! > (n+1)n! = (n+1)!$ ☺

(5)

$$\underline{\begin{bmatrix} 3 & 3 & 0 & -2 \\ 2 & 2 & -1 & 0 \end{bmatrix}} \to \begin{bmatrix} 1 & 1 & 0 & -2/3 \\ 2 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -2/3 \\ 0 & 0 & -1 & 4/3 \end{bmatrix} \to \underline{\begin{bmatrix} 1 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & -4/3 \end{bmatrix}}$$

↑ free variable

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\begin{bmatrix} -y-2/3 \\ y \\ -4/3 \end{bmatrix}} = \begin{bmatrix} -2/3 \\ 0 \\ -4/3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

(6)

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 1 \end{bmatrix} \to$$

← det = −2

↑ volumes doubled

$$\to \begin{bmatrix} 1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & -1/2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1/2 & -3/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & -1/2 & -1/2 \end{bmatrix} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & -3/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & -1/2 & -1/2 \end{array}\right]$$

← A^{-1}

$$X = A^{-1}B = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ -1/2 & -3/2 & -1/2 \\ -1/2 & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 0 & 0 & -1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}}$$

7 a) $\det(A-\lambda I) = \det\begin{bmatrix} -4-\lambda & 13 \\ 7 & 2-\lambda \end{bmatrix} = (-4-\lambda)(2-\lambda) - 7\cdot 13 =$

$= \lambda^2 + 2\lambda - 99 = (\lambda+11)(\lambda-9)$ ∴ Eigenvalues: $\boxed{-11, 9}$

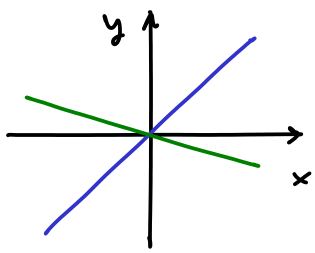
b) $A + 11I = \begin{bmatrix} 7 & 13 \\ 7 & 13 \end{bmatrix} \to \boxed{7x+13y=0}$

$A - 9I = \begin{bmatrix} -13 & 13 \\ 7 & -7 \end{bmatrix} \to \boxed{x=y}$

eigenvectors: $\boxed{\begin{bmatrix} 13 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

c) Let $P = \begin{bmatrix} 13 & 1 \\ -7 & 1 \end{bmatrix}$, then $P^{-1}AP = \frac{1}{20}\begin{bmatrix} 1 & -1 \\ 7 & 13 \end{bmatrix}\begin{bmatrix} -143 & 9 \\ 77 & 9 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & 9 \end{bmatrix}$ ☺

d)



Stretch by a factor of 11 along $\begin{bmatrix} 13 \\ -7 \end{bmatrix}$ and a flip

Stretch by 9 along the main diagonal.

8 First seat the 3 women. Division rule gives $3!/3 = 2$.
Now seat the 6 men into 6 available seats: $2\cdot 6! = \boxed{1440}$

9 Let F = spam, E = hair aug. By Bayes theorem

$$p(F|E) = \frac{p(E|F)\,p(F)}{p(E|F)\,p(F) + p(E|\bar{F})\,p(\bar{F})} = \frac{.14\cdot .55}{.14\cdot .55 + .02\cdot .45} = .895\ldots = \boxed{89.5\%}$$

10 $-1 \pm 4 = \mu \pm \frac{4}{5}\sigma$ so we get $\operatorname{erf}\left(\frac{4/5}{\sqrt{2}}\right) = \operatorname{erf}\left(\frac{2\sqrt{2}}{5}\right) = .576\ldots = 57.6\%$

Have a great summer!