CS3333 Final Fall 2013

(1) Euclid's algorithm:

$70 = 2\cdot 27 + 16$
$27 = 16 + 11$
$16 = 11 + 5$
$11 = 2\cdot 5 + 1$ ← gcd(70,27)

Remainders:

$16 = 70 - 2\cdot 27$
$11 = 27 - 16$
$5 = 16 - 11$
$1 = 11 - 2\cdot 5$

Backsubstitute: $1 = 11 - 2\cdot 5 = 11 - 2(16-11) = 3\cdot 11 - 2\cdot 16$
$= 3\cdot(27-16) - 2\cdot 16 = 3\cdot 27 - 5\cdot 16 = 3\cdot 27 - 5(70 - 2\cdot 27)$
$= \underbrace{13\cdot 27}_{351} - \underbrace{5\cdot 70}_{350}$ Bézout coefficients: $-5, 13$

(2) Suppose $a \equiv a' \bmod m$, $b \equiv b' \bmod m$. Then $\exists j,k$ $a-a'=jm$, $b-b'=km$, so $(a+b)-(a'+b') = a-a'+b-b' = jm+km = (j+k)m$, so $a+b \equiv a'+b' \bmod m$.

(3) $2x \equiv 5 \bmod 7 \iff 2x \equiv 12 \bmod 7 \iff x \equiv 6 \bmod 7$
$3x \equiv 7 \bmod 11 \iff 3x \equiv 18 \bmod 11 \iff x \equiv 6 \bmod 11$

$M = 5\cdot 7\cdot 11 = 385$

i	$M_i = \frac{M}{m_i}$	mod m_i	$y_i = M_i^{-1} \bmod m_i$	b_i = r.h.s.	$M_i y_i b_i$	mod M
1	77	2	3	2	462	77
2	55	6	6	6	1980	55
3	35	2	6	6	1260	105
					Sum:	237

$x \equiv 237 \bmod 385$

(4)

n	n^2	$n!$
0	0	1
1	1	1
2	4	2
3	9	6
4	16	24
5	25	120

Claim: For $n \geq 4$ $n^2 \leq n!$

Basis of induction: $4^2 \leq 4!$ ☺

Suppose $n \geq 4$ and $n^2 \leq n!$

We want: $(n+1)^2 \leq (n+1)!$, i.e. $n+1 \leq n!$

Since $n^2 \leq n!$, it's enough to show $n+1 \leq n^2$

Proof $n \geq 4 \Rightarrow n-1 \geq 3 \Rightarrow n(n-1) \geq 1$
$\Rightarrow n^2 - n \geq 1 \Rightarrow n+1 \leq n^2$ ☺

$\therefore n^2 \leq n!$ for all $n \geq 0$, except 2 and 3.

(5) $f(n) = 2(-3)^n$ for $n \geq 0$

Basis of induction: $n=0$ $\quad f(0) = 2(-3)^0 = 2$

If $n>0$ and $f(n-1) = 2(-3)^{n-1}$, then

$f(n) = -3f(n-1) = -3\cdot 2(-3)^{n-1} = 2(-3)^n$

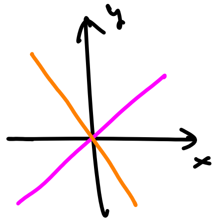
(6) $\begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 4 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$. cycle rows $(1,2,3)$: $\left[\begin{array}{ccc|c} 4 & 3 & 0 & 3 \\ 0 & 2 & 1 & 4 \\ 3 & 0 & 1 & 2 \end{array}\right]$. Scale rows 1, 2, then $r_3 \mapsto r_3 - 3r_1$

$$\begin{bmatrix} 1 & \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & -\frac{9}{4} & 1 & -\frac{1}{4} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{17}{8} & \frac{17}{4} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \boxed{0} \\ 0 & 1 & 0 & \boxed{1} \\ 0 & 0 & 1 & \boxed{2} \end{bmatrix}$$

$\therefore x_1 = 0,\ x_2 = 1,\ x_3 = 2$ $\qquad$ det $= 4 \cdot 2 \cdot \frac{17}{8} = 17$

(7) $A - \lambda I = \begin{bmatrix} 4-\lambda & 5 \\ 6 & 2-\lambda \end{bmatrix}$. $\det(A-\lambda I) = (4-\lambda)(2-\lambda) - 6\cdot 5$

$= 8 - 6\lambda + \lambda^2 - 30 = \lambda^2 - 6\lambda - 22$. Solve: $\frac{6 \pm \sqrt{36 - 4\cdot 22}}{2} = 3 \pm \sqrt{31}$



$\lambda_1 = 3 + \sqrt{31}$ $\quad \lambda_2 = 3 - \sqrt{31}$ $\qquad A - \lambda_1 I = \begin{bmatrix} 1-\sqrt{31} & 5 \\ 6 & -1-\sqrt{31} \end{bmatrix}$

≈ 8.567764363 $\quad \approx -2.567764363$

$(1-\sqrt{31})x_1 + 5x_2 = 0$ $\qquad x_2 = \frac{-1+\sqrt{31}}{5}x_1$ $\qquad \bar{v}_1 = \begin{bmatrix} 5 \\ -1+\sqrt{31} \end{bmatrix}$

slope $\approx .9135528730$

Similarly for λ_2 $\quad x_2 = \frac{-1-\sqrt{31}}{5}x_1$ $\qquad \bar{v}_2 = \begin{bmatrix} 5 \\ -1-\sqrt{31} \end{bmatrix}$

slope ≈ -1.313552873

Stretch by ≈ 8.6 along $\bar{v}_1$, flip and stretch by ≈ -2.6 along $\bar{v}_2$

(8) $n!$ arrangements of pairs. For each pair choose left vs. right: 2^n

By the product rule the total is $n!\,2^n$. $\qquad 4!\,2^4 = 384$.

(9) dice: $n=4$, points: $k = \underset{\text{total}}{8} - \underset{\text{min}}{4} = 4$. $C(n+k-1, k) = C(7,4) = \frac{7!}{3!\,4!} = 35$

Total: $6^4 = 1296$ $\qquad$ Prob: $35/1296 \approx .02700617284 \approx 2.7\%$

(10) If $x - \mu = \pm 2$, $z = \frac{x-\mu}{\sigma} = \pm\frac{2}{6} = \pm\frac{2}{3}$

$P\left(-\frac{2}{3} < z < \frac{2}{3}\right) = 2P\left(0 < z < \frac{2}{3}\right) = 2\cdot\frac{1}{2}\,\mathrm{erf}\left(\frac{2/3}{\sqrt{2}}\right) = \mathrm{erf}\left(\frac{\sqrt{2}}{3}\right)$

$\approx .4950149247 \approx 49.5\%$