Midterm 1 / 2004.3.3 / Algebra II / MAT 5313

Name: ______________________________

Please show all work and justify your answers.

Let K denote a commutative ring.

1. Suppose F is free K-module on 2 generators. Prove from first principles that the set $\operatorname{Alt}_2 F$ of all bilinear alternating maps $F^2 \to K$ is a free cyclic K-module.

2. Let F be a vector space (a free module) over a field K and $L\colon F \to F$ a linear map (a K-module morphism). A scalar λ is called an *eigenvalue* of L whenever the map $\lambda I - L$, fails to be bijective ($I : F \to F$ denotes the identity map, so λI is the dilation by λ). Assuming F is finitely generated, prove that the set of all eigenvalues of L is finite.

   Hint: You may want to represent linear transformations by matrices.

3. Let A be a K-module and define $f\colon A \times K^2 \to A^2$ by $f(a, (\kappa, \lambda)) = (\kappa a, \lambda a)$.
   (a) Prove that f is bilinear.
   (b) Prove that f is universal among bilinear maps on $A \times K^2$ by showing that for any bilinear $g\colon A \times K^2 \to C$ there exists unique linear $g'\colon A^2 \to C$ with $g = g' \circ f$.

   Hint: Write $(a, b) = (a, 0) + (0, b)$. What elements does f take to the two summands?

4. Compute the following tensor products over $\mathbf{Z}$.
   (a) $\mathbf{Z}^2 \otimes \mathbf{Z}^3 \cong ?$
   (b) $\mathbf{Z}^2 \otimes \mathbf{Z}_3 \cong ?$
   (c) $\mathbf{Z}_2 \otimes \mathbf{Z}_3 \cong ?$
   (d) $\mathbf{Z}^2 \otimes \mathbf{Q} \cong ?$
   (e) $\mathbf{Z}_2 \otimes \mathbf{Q} \cong ?$

1	2	3	4	total (40)	%

Prelim. course grade: %