Algebra II, MAT 5313

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Name: ____________________ Pseudonym: ____________________

1. (20 pts.) Suppose R and S are rings and $f\colon R \to S$ is a ring homomorphism.
   (a) Show that $\ker f = \{r \in R\colon f(r) = 0\}$ is a two-sided ideal of R.
   (b) Show that if f is onto and I is a left ideal of R,
   then $f(I) = \{s \in S\colon \exists r \in R\ f(r) = s\}$ is a left ideal of S.

2. (50 pts.) Suppose R is a commutative ring with 1. For each of the following subsets of R prove or disprove that it is closed under multiplication:
   (a) The set of units of R.
   (b) The set of nonunits of R.
   (c) The set of nonzero elements of R.
   (d) A prime ideal.
   (e) The complement of a prime ideal.

3. (60 pts.) Let R be the ring of all continuous real valued functions of a real variable, i.e. $R = \{f\colon \mathbf{R} \to \mathbf{R}\colon f \text{ is continuous}\}$, where addition and multiplication of functions are pointwise, i.e. $(f+g)(x) = f(x)+g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$.

   Given a subset of the real line $V \subseteq \mathbf{R}$ define $I(V)$ to be the set of all continuous functions that vanish on V, i.e. $I(V) = \{f \in R\colon \forall x \in V\ f(x) = 0\}$.
   (a) Which functions are the units of R?
   (b) Prove or disprove: R is an integral domain.
   (c) Show that if $V \subseteq \mathbf{R}$, then $I(V)$ is an ideal of R.
   (d) What are $I(\emptyset)$ and $I(\mathbf{R})$?
   (e) Show that if $a \in \mathbf{R}$, then $I(\{a\})$ is a prime ideal of R.
   (f) Show that if $a, b \in \mathbf{R}$ and $a \neq b$, then $I(\{a, b\})$ is not a prime ideal of R.

4. (40 pts.) True/false questions. Justification (proof or counterexample) required.

T F (a) Every finite integral domain is a field.
T F (b) If R is an integral domain and $S = R \setminus \{0\}$, then $S^{-1}R$ is a field.
T F (c) Every ideal of $\mathbf{Z}$ is a principal ideal.
T F (d) Every ideal of the polynomial ring $\mathbf{Z}[x]$ is a principal ideal.

1		2					3						4				Total	
a	b	a	b	c	d	e	a	b	c	d	e	f	a	b	c	d	(170)	%