Theory of Functions of a Complex Variable II, MAT 5233
Final, May 5, 1997
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Name: ____________________ Pseudonym: ____________________

Throughout, unless otherwise indicated, assume that $\mathbf{C}$ is the complex plane; Ω is a lattice in $\mathbf{C}$; D is a domain in $\mathbf{C}$; and Σ is the Riemann sphere. Show all work.

1. (20 pts.) Suppose $f\colon \mathbf{C}/\Omega \to \Sigma$ is a nonconstant elliptic function. Prove that
   (a) f has at least one pole.
   (b) If f has exactly one pole, then it cannot be a simple pole.
   (Hint: Integrate df/f around a fundamental parallelogram of Ω.)

2. (30 pts.) Let $f(z) = \sum_{\omega\in\Omega} (z-\omega)^{-5}$.
   (a) Show that the above series for $f(z)$ converges normally on $\mathbf{C}\setminus\Omega$. You may use the fact that $\sum_{\omega\in\Omega\setminus\{0\}} |\omega|^{-5}$ converges.
   (b) What are the poles of $f(z)$ and what is their multiplicity?
   (c) Prove that $f(z)$ is elliptic.

3. (10 pts.) Let $\mathcal{P}(z) = z^{-2} + \sum_{\omega\in\Omega\setminus\{0\}} \left((z-\omega)^{-2} - \omega^{-2}\right)$.
   Suppose $\mathcal{P}(z_1) = \mathcal{P}(z_2)$. Prove that $z_1 \pm z_2 \in \Omega$.

4. (10 pts.) Prove that the unit circle is the natural boundary for $\sum_{k=0}^{\infty} z^{k!}$.

5. (10 pts.) Find two paths in $\mathbf{C}\setminus\{0\}$ with the same endpoints such that the meromorphic continuations of $\log z$ along these paths differ by $4\pi i$.

6. (20 pts.) True or false — circle your choice. No justification necessary.

T F (a) A series cannot be meromorphically continued through at least one point on the boundary of its disk of convergence.

T F (b) The Riemann surface of $\sqrt{z^2-1}$ is a torus.

T F (c) The Riemann surface of $\sqrt[3]{z}$ is compact.

T F (d) $1/\log z$ is holomorphic at 0.

1	2	3	4	5	6	total (100)