Final exam / 2002.12.11 / Theory of Functions of a Complex Variable I / MAT 5223.001

Name: ____

Please show all work and justify your answers.

1. (10 pts.) Sketch the sets $H = \{z \in \mathbf{C}\colon \Re[z] \geq \Im[z]\}$ and $D = \{z \in \mathbf{C}\colon |z - i| \geq 2\}$. Find a fractional linear transformation that maps H onto D. Is such a fractional linear transformation unique? Explain.

   Hint: first map both H and D to the upper half plane

2. (10 pts.) Let $D = \{z \in C\colon |z| < 1\}$. Suppose $f \in \mathcal{H}(D)$ and let $g(z) = \overline{f(\overline{z})}$. Prove or disprove that $g \in \mathcal{H}(D)$.

3. (10 pts.) Define $f\colon \mathbf{C} \to \mathbf{C}$ by $f(z) = \overline{z}$. Prove or disprove that f is conformal.

4. (10 pts.) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2z - i)^n}{n}$

   (a) Show that this is a power series. Determine and sketch the disc of convergence.

   (b) Does the series converge on the boundary of the disc of convergence? Explain.

5. (20 pts.) Sketch the given paths γ and evaluate the following integrals along them:

   (a) $\displaystyle\int_\gamma \frac{dz}{3z^3 + iz^2}$, where $\gamma = \{z \in \mathbf{C}\colon |z| = 0.1\}$

   (b) $\displaystyle\int_\gamma \Im[z]\, dz$, where γ is the straight line segment from -1 to $-i$

6. (10 pts.) Prove that $\displaystyle\left|\int_0^1 \sqrt{t} e^{it}\, dt\right| \leq \frac{2}{3}$

7. (10 pts.) Suppose $\Omega \subseteq \mathbf{C}$ is a domain and f_n is a sequence in $\mathcal{H}(\Omega)$ such that $f_n \to f$ uniformly on compact subsets of Ω. By the theorem of Weierstrass $f \in \mathcal{H}(\Omega)$. Prove that for any $z \in \Omega$ we have $f_n'(z) \to f'(z)$.

   Hints: Cauchy integral formula (which path?); uniform convergence $\Rightarrow$ termwise integration

8. (20 pts.) Let $D = \{z \in \mathbf{C}\colon |z| < 1\}$. Suppose $\Omega \subseteq \mathbf{C}$ is a domain and $\overline{D} \subseteq \Omega$. Let $f \in \mathcal{H}(\Omega)$ such that $f(0) = 0$.

   (a) Prove that there exists $g \in \mathcal{H}(\Omega)$ such that $f(z) = z\,g(z)$. Determine $g(0)$.

   (b) Show that the maxima of $|f|$ and $|g|$ on $\overline{D}$ exist and are equal.

   Hints: (a) Taylor series, principle of analytic continuation; (b) maximum modulus principle

1	2	3	4	5	6	7	8	total (100)
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