Theory of Functions of a Complex Variable I / MAT 5223.001
Final / December 9, 1998 / Instructor: D. Gokhman

Name: _____ Pseudonym: _____

Please show all work.

1. (5 pts.) Find and sketch all $z \in \mathbf{C}$ such that $z^3 = 8i$.

2. (10 pts.) Prove that $\{z \in \mathbf{C}\colon 1 < |z| < 2\}$ is open in $\mathbf{C}$. Sketch this set.

3. (10 pts.) Given $z, w \in \mathbf{C} \setminus \{0\}$, construct a path $\gamma : [a, b] \to \mathbf{C} \setminus \{0\}$ such that $\gamma(a) = z, \gamma(b) = w$. What can you conclude about $\mathbf{C} \setminus \{0\}$?

4. (10 pts.) Suppose $U \subseteq \mathbf{C}$ is open and $K \subseteq U$ is compact. Prove that $\exists \varepsilon > 0$ and $z_k \in K, 1 \le k \le n$ such that $K \subseteq \bigcup_{k=1}^{n} \{z\colon\ |z - z_k| < \varepsilon\} \subseteq U$.

5. (10 pts.) For each of the following $f : \mathbf{C} \to \mathbf{C}$ find the set where f is complex differentiable and the set where f is analytic.

$$\text{(a) } f(z) = \frac{1}{\sin z} \qquad \text{(b) } f(z) = \cos^2 \overline{z}$$

6. (10 pts.) Let $f(z) = z^8/(2 - z)^2$.
   (a) Expand $f(z)$ in a power series at the origin.
   (b) Find the radius of convergence of this power series.

7. (20 pts.) Evaluate the following integrals.
   (You may use the Cauchy Integral Formula, where applicable.)
   (a) $\displaystyle\int_\gamma \frac{dz}{2iz - 1}$, where γ is the unit circle traversed counterclockwise,
   (b) $\displaystyle\int_\gamma \frac{dz}{\overline{z}}$, where γ is the unit circle traversed counterclockwise,
   (c) $\displaystyle\int_\gamma \frac{dz}{z^3 + 2iz^2}$, where γ is the unit circle traversed counterclockwise,
   (d) $\displaystyle\int_\gamma \operatorname{Im} z\, dz$, where γ is the segment from $-i$ to -1.

8. (10 pts.) Suppose $f\colon \mathbf{C} \to \mathbf{C}$ is analytic and $\exists a \in \mathbf{R} \quad \forall z \in \mathbf{C} \quad |f(z)| \le a\,|z|^2$. Prove that $\exists c \in \mathbf{C}$ such that $\forall z \in \mathbf{C}\ f(z) = cz^2$.

1	2	3	4	5	6	7	8	total (85)	%