Midterm 1 / 2013.10.10 / MAT 5173.001 / Algebra I

Name: ____

Please show all work and justify your answers.

1. Show that if G is a group, the natural projection $G \to G/[G,G]$, where $[G,G]$ is the commutator subgroup, is universal among homomorphisms from G to Abelian groups.

2. Suppose G, H are normal subgroups of a group K such that $G \vee H = K$ and $G \cap H = e$. Prove that $K \cong G \times H$.

   Hint: Show that elements of G commute with those of H. Then define $\theta\colon G \times H \to K$ by $\theta([g,h]) = gh$.

3. Let $g \in \mathbf{Q}[x]$. For $p \in \mathbf{Q}[x]$ let $E_g(p) = p(g)$. Show that any endomorphism of $\mathbf{Q}[x]$ is of this form. For which g is E_g an automorphism?

4. For r in a ring and $n \in \mathbf{N}$, let nr denote the sum of r with itself n times. For a ring R and $n \in \mathbf{N}$, let $nR = \{nr\colon r \in R\}$.

   (a) Prove nR is an ideal of R.

   (b) Construct a functor from the category of rings to itself, that on objects takes R to R/nR, by finding an appropriate definition for how the functor acts on morphisms.

   Hint: Use the universal property of quotient rings.

   (c) Prove that this indeed defines a functor.

5. Suppose k is a field and $c \in k$. Prove that the ideal generated by $x - c$ in the polynomial ring $k[x]$ is a maximal ideal.

1	2	3	4	5	total (50)	%

Prelim. course grade: %