Final Exam / 2003.12.9 / Algebra I / MAT 5173

Name: ____________________

Please show all work and justify your answers.

1. How many group endomorphisms does $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ have? Exhibit all group automorphisms of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. What famous group is $\mathrm{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ isomorphic to? Explain.

2. Consider the additive group $\mathbf{R}^2$. Let H be the subgroup generated by $\{(1,3),(3,1)\}$. Describe all cosets of H. Sketch H and a typical coset different from H.

3. Let $GL_2(\mathbf{R})$ denote the group of all invertible 2×2 real matrices under multiplication and let $SL_2(\mathbf{R}) = \{A \in GL_2(\mathbf{R})\colon\ |\det(A)| = 1\}$.

   (a) Prove that $SL_2(\mathbf{R})$ is a subgroup of $GL_2(\mathbf{R})$.

   (b) Prove that $H = \{A \in SL_2(\mathbf{R})\colon\ \det(A) = 1\}$ is a normal subgroup of $SL_2(\mathbf{R})$.

   (c) Exhibit, with proof, a subgroup of $SL_2(\mathbf{R})$ that is not normal.

4. How many group morphisms $\mathbf{Z}_2 \to \mathbf{Z}_8$ are there? Ring morphisms? Explain.

5. How many ring automorphisms of $\mathbf{Z}_3[x]$ are there? Explain.

6. Suppose K is an integral domain. Let $a \in K$ and let J be the ideal of $K[x]$ generated by $x - a$.

   (a) Prove that $K[x]/J \cong K$.

   (b) Prove or disprove: J is a maximal ideal of $K[x]$.

7. Construct the coproduct of two R-modules A and B by specifying the object and the two insertions. State the universal property of coproduct and prove that your construction satisfies it.

8. (a) Prove that every subgroup of $\mathbf{Z}$ is a free $\mathbf{Z}$-module (i.e. a free abelian group).

   (b) Give an explicit example of a $\mathbf{Z}$-module that is not free. Explain.

9. Suppose $\varphi\colon A \to B$ is an R-module epimorphism and $\varphi^*\colon B^* \to A^*$ is its dual. Prove that $\mathrm{Im}\,(\varphi^*) = \{f \in \mathrm{Hom}_R[A,R]\colon\ f_*(\ker\varphi) = 0\}$

10. Prove that $R^2 \cong (R^2)^*$.

1	2	3	4	5	6	7	8	9	10	total (100)