Final exam / 2022.5.11 / MAT 4273.001 / Topology

Name: ____

Please show all work and justify your answers.

1. Suppose X and Y are topological spaces and $p\colon X \times Y \to X, q\colon X \times Y \to Y$, $p(x, y) = x, q(x, y) = y$ are the canonical projections from the product.

   (a) Prove that p and q are open maps (forward images of open sets are open).

   (b) Provide a concrete example to show that p and q do not necessarily carry closed sets to closed sets.

2. Define an equivalence relation on the cylinder $S^1 \times I$ by $[e^{i\theta}, t] \sim [e^{i\theta'}, t'] \Leftrightarrow t = t' = 0$ or $t = t' = 1$ or $\theta = \theta'$ and $t = t'$. Prove that the quotient space $(S^1 \times I)/\sim$ is homeomorphic to S^2.

3. Prove that two finite discrete topological spaces are homotopy equivalent if, and only if, they have the same number of elements.

4. Suppose X is a topological space. Recall that a path in X is a continuous map from the unit interval I to X. Prove that

   (a) if $x_0, x_1 \in X$ and there is a path $s\colon I \to X$ with $s(0) = x_0$, $s(1) = x_1$, then there is a path $s'\colon I \to X$ with $s'(0) = x_1$, $s'(1) = x_0$

   (b) if $p, q\colon I \to X$ are paths such that $p(1) = q(0)$, then there is a path $r\colon I \to X$ such that $r(0) = p(0), r(1) = q(1)$

   (c) if X is contractible (homotopy equivalent to a singleton), then X is path connected (and thus connected).

   Hint: First prove that if $H\colon X \times I \to X$ is a homotopy, then for each $x \in X$ the restriction of H to $\{x\} \times I$ gives a path in X. Then use parts (a), (b).

1	2	3	4	total (40)