1. Suppose G is a group.

   (a) If $a \in G$, $n \in \mathbf{N}$, and $a^n = e$, prove that the order $|a|$ divides n.

   (b) Suppose $a, b \in G$ commute, have finite orders and the subgroups they generate are essentially disjoint. Prove that $|ab| = \operatorname{lcm}(|a|, |b|)$.

   Definitions: Any $x \in G$ generates a cyclic subgroup $\langle x \rangle = \{x^n \colon n \in \mathbf{Z}\}$. If H and K are subgroups of G, we say H and K are are essentially disjoint when $H \cap K = \{e\}$.

2. For each $k \in \mathbf{Z}$ define the set of all integer multiples $\langle k \rangle = k\mathbf{Z} = \{kn \colon n \in \mathbf{Z}\}$.

   (a) Prove that $\langle k \rangle$ is a subgroup of $\mathbf{Z}$

   (b) If H is a nontrivial subgroup of $\mathbf{Z}$, prove that there exists $k \in \mathbf{N}$ such that $H = \langle k \rangle$

   Hint: Consider the set of all positive elements of H.

3. Suppose $\varphi \colon \mathbf{Z}_{35} \to \mathbf{Z}_{14}$ is a group homomorphism and $\varphi(4) = 12$.

   (a) Find a formula for φ.

   (b) Find the image and the kernel of φ. What are the orders of these subgroups and how are these orders related? Explain.

   Hint: Fibers of φ partition G. How big is each nonempty fiber and how many of them are there?