Final exam / 2019.5.9 / MAT 4233.001 / Modern Abstract Algebra

Name: ____

Please show all work. If you use a theorem, name it or state it.

1. Suppose m and n are natural numbers. Prove that
   (a) any common divisor of m and n divides $\gcd(m, n)$.
   (b) $\operatorname{lcm}(m, n)$ divides any common multiple of m and n.
2. Sketch the subgroup lattice for $\mathbf{Z}_{28}$. For each subgroup, list all the elements and indicate all possible generators of the subgroup.
3. Find a proper non-trivial normal subgroup of the symmetric group S_n. Find a subgroup of S_n that is not normal. Prove your assertions.
4. Suppose G is a finite group with m elements and $x \in G$. Prove:
   (a) x has finite order.
   (b) $x^n = e$ if and only if the order of x divides n.
   (c) $x^m = e$.
5. Let R be the ring of continuous functions $\mathbf{R} \to \mathbf{R}$ with pointwise operations. Define $\varepsilon\colon R \to \mathbf{R}^2$ by $\varepsilon(f) = [f(0), f(1)]$. Prove that ε is a ring homomorphism. Is ε onto? Is $\ker \varepsilon$ a maximal ideal? Prime ideal?
6. Suppose $m, n, k \in \mathbf{N}$ with $\operatorname{lcm}(m, n) = k$. Define a group homomorphism $\varphi\colon \mathbf{Z} \to \mathbf{Z}_m \oplus \mathbf{Z}_n$ by $\varphi(i) = [i \bmod m, i \bmod n]$. Prove that $\ker \varphi = k\mathbf{Z}$. What does the first isomorphism theorem tell you about the image of φ? What can you say about $\mathbf{Z}_m \oplus \mathbf{Z}_n$ if $\gcd(m, n) = 1$?
7. Show that the set of all polynomials in $\mathbf{Z}[x]$ with even constant term is a maximal ideal of $\mathbf{Z}[x]$. What is the quotient ring?

1	2	3	4	5	6	7	total (70)