Final exam / 2017.12.11 / MAT 4233.001 / Modern Abstract Algebra

Name: ____

Please show all work. If you use a theorem, name it or state it.

1. Let $m \in \mathbf{N}$ and $m\mathbf{Z} = \{mk\colon k \in \mathbf{Z}\}$. Prove $m\mathbf{Z}$ is an ideal of $\mathbf{Z}$. Conversely, prove that any ideal of $\mathbf{Z}$ is of this form.
2. Suppose $\alpha = (1,6,2,5,3)(4,7,3,5,1,2)(2,6)$ is a permutation (in cycle notation). What is the order of α? What is the parity of α? Express α^{404} as a product of disjoint cycles.
3. Prove that the set of all rotations in the dihedral group D_n is a normal subgroup of D_n. Exhibit a subgroup of D_4 that is not normal. Explain.
4. How many group homomorphisms are there from $\mathbf{Z}$ to $\mathbf{Z}_{40}$? How many of them are one-to-one? How many of them are onto?
5. Suppose G is finite group of order n and $a \in G$. Prove that $a^n = e$. What can you conclude about the order of a, if n is prime? What can you conclude about groups of prime order?
6. Let $\mathbf{C}^*$ denote the multiplicative group of nonzero complex numbers. Define $\varphi\colon \mathbf{R} \to \mathbf{C}^*$ by $\varphi(t) = e^{2\pi i t}$. Prove that φ is a group homomorphism. What are its kernel and image? What conclusion can you draw from the First Isomorphism Theorem?
7. Suppose F is a field and p is polynomial in $F[x]$ of degree 2 or 3. Prove that p irreducible if and only if p has no roots in F. Give an explicit counter example for degree 4.
8. Let J be the ideal generated by x and 3 in $\mathbf{Z}[x]$. Prove that J is a maximal ideal.

1	2	3	4	5	6	7	8	total (80)