Final exam / 2013.5.10 / MAT 4233.001 / Modern abstract algebra

Name: ____________________

Please show all work and justify your answers.

1. Show $\begin{bmatrix} 7 & 5 \\ 4 & 5 \end{bmatrix} \in GL(2, \mathbf{Z}_{17})$ and find its inverse.

2. Determine the subgroup lattice of the dihedral group D_4. Pick a nontrivial subgroup of D_4, except the subgroup of all rotations, and find all its left cosets.

3. Prove that a cyclic group is isomorphic to $\mathbf{Z}$ or $\mathbf{Z}_m$ for some m.

4. Let $G = \mathbf{Z}_4 \oplus \mathbf{Z}_4$, $H = \langle [2, 0], [0, 2] \rangle$, and $K = \langle [2, 3] \rangle$. Find all the cosets of H and K in G. Determine the isomorphism classes of G/H and G/K as finite abelian groups, i.e. represent each group as an external product of $\mathbf{Z}_m$'s.

5. Suppose G is a finite group, $H < G$ and $K \triangleleft G$. Prove that $HK < G$ and $KH < G$.

6. Suppose G is a group of order p^k, where p is a prime and $k \geq 1$. Prove that G contains an element of order p.

7. Let $n \in \mathbf{N}$. Prove that $\langle n \rangle$ is a maximal ideal of $\mathbf{Z}$ if and only if n is prime.

8. Suppose R is a commutative ring with unity and $R \setminus U(R)$ is an ideal. Prove that this ideal is the unique maximal ideal of R. Conversely prove that if M is the unique maximal ideal of R, then $R \setminus M = U(R)$.

1	2	3	4	5	6	7	8	total (80)	%