Final exam / 2011.5.6 / MAT 4233.001 / Modern abstract algebra

Name: ____

Please show all work and justify your answers.

1. Suppose G is a multiplicative group, $a \in G$. Prove that $a^n = e \Leftrightarrow |a|$ divides n.

2. Prove that any group with prime order must be cyclic.

3. How many distinct group automorphisms of $\mathbf{Z}$ are there? Explain. What about $\mathbf{Z}_p$?

   Hint: think about generators and their possible values under an automorphism.

4. Let L be a line in $\mathbf{R}^3$ through the origin and let $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the orthogonal projection to L. Describe $\ker\varphi$ and its cosets in $\mathbf{R}^3$ geometrically. Sketch.

5. Suppose H and K are subgroups of a finite group G and one of them is normal in G. Let HK denote the set of all products $\{hk\colon h \in H, k \in K\}$. Prove that $HK < G$.

   Hint: consider a product $hkh'k'$ and observe that kh' belongs to Kh' and kH.

6. Suppose $\varphi\colon R \to S$ is a homomorphism of rings. Prove that $\ker\varphi$ is an ideal of R. Show by example that $\varphi(R)$ is not necessarily an ideal of S. What hypothesis on φ would ensure that $\varphi(R)$ is an ideal of S? Prove it.

7. Let $R = \mathbf{R}[x]$. Which elements of R are units? Are there nonzero zero divisors in R? Let $A = \{p(x) \in R\colon p(0) = 0\}$. Prove that A is an ideal of R. Is A a prime ideal? Maximal? Prove your assertions.

8. Prove that any cubic polynomial in $\mathbf{R}[x]$ is reducible. Are there irreducible cubic polynomials in polynomial rings in one variable over other fields? Explain.

1	2	3	4	5	6	7	8	total (80)