Real Analysis II, MAT 4223
Final Exam, May 3, 1993
Instructor: D. Gokhman

Show all pertinent work. Prove your assertions. If you use theorems in your proofs, cite them explicitly either by name or by stating them. Answers alone are not sufficient. Box your answers.

Name: ____

1. (40 pts.) Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable. Prove that
   (a) f is continuous,
   (b) f is uniformly continuous, if $|f'|$ is bounded.
2. (40 pts.) Suppose $\alpha\colon [a,b] \to \mathbf{R}$ is monotone increasing. Use upper and lower Darboux–Stieltjes sums to do the following.
   (a) Prove directly from the definition that constants are Riemann–Stieltjes integrable with respect to α on $[a,b]$.
   (b) Find $\int_a^b d\alpha$.
3. (40 pts.) Determine whether the following sequences converge uniformly on $[0,1]$.
$$\text{(a) } \frac{1}{1+nx}, \qquad \text{(b) } (1-x)x^n.$$
4. (40 pts.) Classify all functions $f : \mathbf{R} \to \mathbf{R}$ such that the family $\{f_n\colon n \in \mathbf{N}\}$, where $f_n(x) = f(nx)$, is equicontinuous.
5. (40 pts.) Let $f(x) = |x| - \frac{\pi}{2}$ on $(-\pi, \pi]$.
   (a) Find the coefficients of the corresponding Fourier series $f(x) \sim a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$.
   (b) Find $\sum_{k=1}^{\infty} a_k$.
   (c) Find $\sum_{k=1}^{\infty} (-1)^k a_k$.
   (d) Find $\sum_{k=1}^{\infty} a_k^2$.