University of Texas at San Antonio

Real Analysis II, MAT 4223
Exam $\mathcal{N}^{\underline{o}}$1, 2/26/92
Instructor: D. Gokhman

Name: ______________________________

1. (30 pts.) For the functions $f\colon[-5,5]\to\mathbf{R}$ given below

   (i) sketch f,

   (ii) find $\varphi_f[-5,5]$ (the total variation on $[-5,5]$) and $\omega_f(0)$,

   (iii) for the partition $\{-5,0,5\}$ and $\xi_1=-\pi, \xi_2=1$ calculate the Riemann sum $S(P,f,\xi)$,

   (iv) find $\mathcal{D}_f$ (points of discontinuity) and $m^*(\mathcal{D}_f)$,

   (v) determine whether $f\in\mathcal{R}[-5,5]$ (Riemann integrable).

   (a)
$$f(x)=\begin{cases} |x| & \text{for } x\in\mathbf{Q},\\ -|x| & \text{otherwise.}\end{cases}$$

   (b)
$$f(x)=\begin{cases} 0 & \text{if } x=0,\\ x\cos\left(x^{-1}\right) & \text{otherwise.}\end{cases}$$

2. (35 pts.) True of false questions, circle your choice. If you choose TRUE, give a reason. If FALSE, provide a counterexample.

   Suppose $f\colon[a,b]\to\mathbf{R}$.

T F (a) If f is differentiable, then $f\in\mathcal{R}[a,b]$.

T F (b) If f is defined by
$$f(x)=\begin{cases} 1 & \text{for } x\in\mathbf{Q},\\ 0 & \text{otherwise,}\end{cases}$$
then for any $\varepsilon>0$ there exists a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\varepsilon$.

T F (c) If $a < c < d < b$, then $\varphi_f[c, d] < \varphi_f[a, b]$.

T F (d) If P is a partition of $[a, b]$ and P^* is a refinement of P, then $U(P^*, f) - L(P^*, f) \leq U(P, f) - L(P, f)$.

T F (e) If $f \in \mathcal{R}[a, b]$ and $f > 0$, then $1/f \in \mathcal{R}[a, b]$.

3. (35 pts.) Suppose $f\colon [a, b] \to \mathbf{R}$ is differentiable and $|f'| \leq k$ for some $k > 0$.

   (a) Use the mean value theorem to show that if $[c, d] \subseteq [a, b]$, then $\varphi_f[c, d] \leq k(b - a)$.

   (b) Show that for any partition P of $[a, b]$, $U(P, f) - L(P, f) \leq k(b - a)^2$.