Final exam / 2016.12.14 / MAT 4213.001 / Real Analysis I

Name: ______________________________

Please show all work. If you use a theorem, name it or state it.

1. Suppose $A = \left\{ \dfrac{(-1)^n n}{n+1} : n \in \mathbf{N} \right\}$ and $f\colon A \to \mathbf{R}$ is a bounded function.
   (a) Find all cluster points of A (with proof).
   (b) State the definition of limit and use it to prove that $(x-1)f(x) \to 0$ as $x \to 1$.
2. Prove that $\cos\frac{1}{x}$ fails to have a limit as $x \to 0$, while $\lim\limits_{x\to 0} x\cos\frac{1}{x} = 0$.
3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and $A \subset \mathbf{R}$ is closed and bounded. Prove that the image $f_*(A)$ is closed in $\mathbf{R}$ and bounded.
4. Suppose $f\colon \mathbf{Q} \to \mathbf{R}$ can be extended continuously to $\mathbf{R}$. Prove that such an extension is unique.
5. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is increasing and $c \in \mathbf{R}$. Prove that f has a right limit at c.
6. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 \cos\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable and f' is not continuous at 0.
7. Find the limits at 0 and ∞ of $(1+\frac{1}{x})^x$ and prove your results.
8. Let $f : (0,2) \to \mathbf{R}$ be defined by $f(x) = \ln x$. Let $p_n(x)$ denote the degree n Taylor polynomial for f at 1. Find p_3 and prove that $p_2(x) \le f(x) \le p_3(x)$ for all $x \in (0,2)$.

1	2	3	4	5	6	7	8	total (80)