Final exam / 2010.12.10 / MAT 3233.001 / Modern Algebra

Name: ______________________________

Please show all work.

1. Use induction to show that for $n \geq 1$ the partial sum
$$1^3 + 2^3 + ... + n^3 = \sum_{k=1}^{n} k^3$$
can be expressed in closed form by $\left[\frac{n(n+1)}{2}\right]^2$

2. Use Euclid's algorithm to find $(48, 22)$ and $s, t \in \mathbf{Z}$ such that $(48, 22) = 48s + 22t$.

3. Compute 3^{21} modulo 9 by repeated squaring and reduction. Show work.

4. Suppose R is a commutative ring (with unity) and let U be the set of all units in R.
   (a) Prove that U is a multiplicative group.
   (b) Prove that U cannot contain zero divisors.
   (c) Describe U for the ring $\mathbf{Z}_m$ and the polynomial ring $\mathbf{R}[x]$.

5. Partition U_{17} into cosets of $\langle 13 \rangle$.

6. Consider the set permutations on n elements $\{1, 2, ...n\}$ (with $n \geq 2$) that keep the element 1 fixed: $H = \{\sigma \in S_n \colon \sigma(1) = 1\}$. Prove that H is a subgroup of S_n and express the set of permutations that take 1 to 2: $K = \{\sigma \in S_n \colon \sigma(1) = 2\}$ as a coset of H.

7. Prove that among the residues modulo m it is exactly those that are coprime to m that are units in the ring $\mathbf{Z}_m$.

8. Find the solution set for the system of congruences
$$5x \equiv 2 \bmod 48$$
$$7x \equiv 22 \bmod 30$$

9. Exhibit (with proof) a surjective group homomorphism from the general linear group of invertible linear operators on the real plane under composition (or equivalently, 2×2 nonsingular matrices with real coefficients under matrix multiplication) $\mathrm{GL}_2(\mathbf{R})$ to the multiplicative group of nonzero real numbers $\mathbf{R}^*$. What is this homomorphism's kernel?

1	2	3	4	5	6	7	8	9	total (90)	%