Midterm 2 / 2015.11.24 / MAT 3013.002 / Foundations of Mathematics

Name: ____

Please show all work.

1. Let $f\colon X \to Y$ be a function and for $x, x' \in X$ define $x \sim x' \Leftrightarrow f(x) = f(x')$. Prove $\sim$ is an equivalence relation on X. Describe the equivalence classes for the cases when
   (a) f is injective
   (b) f is constant
   (c) $f\colon \mathbf{Z} \to \mathbf{Z}$, $f(n)$ is the remainder after dividing n by 2.
2. In each case, sketch two distinct equivalence classes, where $\sim$ is as in the above problem.
   (a) $f\colon \mathbf{R}^2 \to \mathbf{R}$, $f([x, y]) = x^2 + y^2$.
   (b) $f\colon \mathbf{R}^2 \to \mathbf{R}$, $f([x, y]) = x - y$.
3. Let $S = \{x \in \mathbf{Q}\colon (\exists n \in \mathbf{N})[x = 1/n]\}$. If they exist, what are $\max S$ and $\min S$? For S as a subset of $\mathbf{Q}$, same question for $\sup S$ and $\inf S$. Prove your assertions.
4. Suppose S is a nonempty set and $\{A_s \subseteq \mathbf{Q}\colon s \in S\}$ is a family of rays to the left each without a maximum. Prove that $\bigcup_{s \in S} A_s$ is a ray to the left without a maximum.

   Definition: $A \subseteq \mathbf{Q}$ is a ray to the left means $x < y \wedge y \in A \Rightarrow x \in A$.
5. Use completeness of $\mathbf{R}$ to show that $\mathbf{N}$ is not bounded above (as a subset of $\mathbf{R}$). Use this to prove the Archimedean property of $\mathbf{R}$: $x, y \in \mathbf{R} \wedge x > 0 \Rightarrow (\exists n \in \mathbf{N})[y < nx]$.

1	2	3	4	5	total (50)