Final exam / 2012.12.18 / MAT 3013.001 / Foundations of Mathematics

Name: ______________________________

Please show all work.

1. Prove that for any positive integer n the quantity $2^n 3^{2n} - 1$ is divisible by 17.
2. Construct a truth table for $(p \to q) \to (q \to p)$. Is it a tautology, contradiction, or neither?
3. Prove that irrational real numbers form a set. You may assume $\mathbf{Z}$ and $\mathbf{R}$ are sets.
4. Prove or disprove subset in each direction between $(A \setminus B) \times (C \setminus D)$ and $(A \times C) \setminus (B \times D)$.
5. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is given by $f(x) = \sin x$. Find the following.
   (a) $f_*(\mathbf{R})$ (b) $f^*(\{1, -1\})$ (c) $A \subseteq \mathbf{R}$ such that $A \neq \varnothing \wedge f^*(A) = \varnothing$
6. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be given by $f([x, y]) = x - y$. Prove that f is surjective and not injective. Find a one-sided inverse for f (with proof). Show that it is not unique.
7. Let $f\colon X \to Y$ be a function and for $x, x' \in X$ define $x \sim x' \Leftrightarrow f(x) = f(x')$. Prove $\sim$ is an equivalence relation on X. Describe the equivalence classes for the case when f is injective. Same for when f is constant. Same for $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = \sin x$.
8. Let $S = \{x \in \mathbf{Q} \colon (\exists n \in \mathbf{N})[x = 3^{-n}]\}$. If they exist, what are $\max S$ and $\min S$? For S as a subset of $\mathbf{Q}$, same question for $\sup S$ and $\inf S$. Prove your assertions about min and inf.
9. Prove that the partial order $\subseteq$ on $\mathbf{R}$ is a linear order, by showing that for any two Dedekind cuts D and D' we have $D \subseteq D' \vee D' \subseteq D$.
10. Show that the intersection of a nonempty family of initial segments in $\mathbf{Q}$ is an initial segment. Give a concrete example of a nonempty collection of Dedekind cuts, whose intersection is not a Dedekind cut.

1	2	3	4	5	6	7	8	9	10	total (100)