Final examination / 2006.5.10 / Calculus II / MAT 1223.005

Name: ______________________________

Please show all work and justify your answers. Make and label sketches. Supply brief narration with your solutions and draw conclusions, including units as appropriate.

1. Find the area inside the circle $r = 1/2$ and outside the cardioid $r = 1 + \cos\theta$. Sketch.

   Hint: Find the polar coordinates of the points of intersection.

2. Determine whether the improper integral $\int_1^\infty \frac{\sqrt{\sqrt{x}+1}}{\sqrt[3]{x^4}}\,dx$ converges. Justify.

3. Demonstrate your mastery of techniques of integration by evaluating the following antiderivatives manually. Show all steps.

$$\text{(a)} \int x^2 \cos(3x)\,dx \qquad \text{(b)} \int \frac{x^2}{x^2+1}\,dx$$

4. A small barn, the shape of a hemisphere of radius 15 m, is filled with hay. The density of hay decreases linearly with height from 250 kg/m^3 at the bottom to 150 kg/m^3 at the top. How far from the ground is the center of mass of the hay?

   Hint: Write down a formula for the density as a function of height. Set up integrals for the total mass of hay and for the vertical moment. You may evaluate integrals on the calculator.

5. A spherical tank with diameter 5 m is half full of water. How much work is involved in pumping all the water out of a small hole at the top of the tank?

   Note: mass density of water $\delta = 10^3$ kg/m^3, gravitational acceleration $g = 9.81$ m/s^2

6. Find the second degree Taylor approximation to $e^x/(1-x)$ near $x = 0$. Sketch the given function and the approximation fairly close to $x = 0$ on the same graph. On which interval would you say the approximation is "good"?

7. Find the second order Fourier approximation to $f(x) = 1 - x$ on the interval $[-1, 1]$. Sketch $f(x)$ and the approximation over the entire interval on the same graph. At what points is the approximation the poorest? Why do you think that is?

8. Suppose $y(x)$ is a solution of the differential equation

$$\frac{dy}{dx} = y\,2^x$$

   satisfying the initial condition $y(0) = 1$. Find $y(1)$.

Fourier series: If f is a continuous function on $(-p/2, p/2)$, then $f(x) = a_0 + \sum_{k=1}^{\infty}[a_k \cos(2\pi kx/p) + b_k \sin(2\pi kx/p)]$, where $a_0 = \frac{1}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\,dx$, $a_k = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\cos(2\pi kx/p)\,dx$, $b_k = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\sin(2\pi kx/p)\,dx$

1	2	3	4	5	6	7	8	total (80)	%