Final exam / 2016.5.5 / CS 3333.002 / Mathematical foundations of computer science

Name: _

Please show all work and justify your answers.

- 1. Let $m \ge 2$. Prove that if $a \equiv b \mod m$, then gcd(a, m) = gcd(b, m). Does the converse hold? Explain.
- 2. Apply the extended Euclidean algorithm to show that 313 and 131 are co-prime and solve $131x \equiv 1 \mod 313$.
- 3. Use the Chinese remainder formula to solve the following system of congruences:

 $x \equiv 2 \mod 5, \qquad x \equiv 5 \mod 6, \qquad x \equiv 3 \mod 7.$

- 4. Prove by induction that $1 + \frac{1}{4} + \frac{1}{9} + \dots \frac{1}{n^2} < 2 \frac{1}{n}$ for n > 1.
- 5. Solve the recurrence $x_n = 2x_{n-1} + 15x_{n-2}$ subject to initial conditions $x_0 = 1, x_1 = 2$. Check your solution by computing x_2 and x_3 using the recurrence relation and comparing the values of x_0, x_1, x_2, x_3 from your formula.
- 6. Solve the linear system y + z = a, x + z = b, x + y = c, where a, b, c are constants.
- 7. Find a matrix A such that $\mathbf{x} \mapsto A\mathbf{x}$ is orthogonal projection of \mathbf{x} to the main diagonal. Same for rotation by 90 degrees. In each case compute A^4 and briefly explain your result geometrically.
- 8. (a) Let $A = \begin{bmatrix} -9 & 8 \\ -12 & 11 \end{bmatrix}$. Find the eigenvalues of A and corresponding eigenvectors.
 - (b) Let S be the matrix whose columns are eigenvectors of A. Verify that $S^{-1}AS$ is diagonal with entries the eigenvalues of A.
 - (c) Sketch the eigenspaces and give a geometrical description of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.
- 9. You are dealt 5 playing cards from a shuffled standard 52 card deck. What are your chances of getting a full house (3 of a kind and a pair)? Two pairs? Explain.
- 10. The likelihood that an outbreak of acne is cured with Biebermax cream t days from onset is given by the probability distribution p(t) = mt + 0.1 for $0 \le t \le 20$ (where m is a constant) and p(t) = 0 for all other t.
 - (a) Find m. On average how soon do people clear up with treatment?
 - (b) If a hundred patients start applying Biebermax, when will half of them clear up?

1	2	3	4	5	6	7	8	9	10	total (100)