Name: $\qquad$
Please show all work and justify your answers.

1. Let $m \geq 2$. Prove that if $a \equiv b \bmod m$, then $\operatorname{gcd}(a, m)=\operatorname{gcd}(b, m)$. Does the converse hold? Explain.
2. Apply the extended Euclidean algorithm to show that 313 and 131 are co-prime and solve $131 x \equiv 1 \bmod 313$.
3. Use the Chinese remainder formula to solve the following system of congruences:

$$
x \equiv 2 \bmod 5, \quad x \equiv 5 \bmod 6, \quad x \equiv 3 \bmod 7 .
$$

4. Prove by induction that $1+\frac{1}{4}+\frac{1}{9}+\ldots \frac{1}{n^{2}}<2-\frac{1}{n}$ for $n>1$.
5. Solve the recurrence $x_{n}=2 x_{n-1}+15 x_{n-2}$ subject to initial conditions $x_{0}=1, x_{1}=2$. Check your solution by computing $x_{2}$ and $x_{3}$ using the recurrence relation and comparing the values of $x_{0}, x_{1}, x_{2}, x_{3}$ from your formula.
6. Solve the linear system $y+z=a, x+z=b, x+y=c$, where $a, b, c$ are constants.
7. Find a matrix $A$ such that $\mathbf{x} \mapsto A \mathbf{x}$ is orthogonal projection of $\mathbf{x}$ to the main diagonal. Same for rotation by 90 degrees. In each case compute $A^{4}$ and briefly explain your result geometrically.
8. (a) Let $A=\left[\begin{array}{rr}-9 & 8 \\ -12 & 11\end{array}\right]$. Find the eigenvalues of $A$ and corresponding eigenvectors.
(b) Let $S$ be the matrix whose columns are eigenvectors of $A$. Verify that $S^{-1} A S$ is diagonal with entries the eigenvalues of $A$.
(c) Sketch the eigenspaces and give a geometrical description of the linear map $\mathbf{x} \mapsto A \mathbf{x}$.
9. You are dealt 5 playing cards from a shuffled standard 52 card deck. What are your chances of getting a full house (3 of a kind and a pair)? Two pairs? Explain.
10. The likelihood that an outbreak of acne is cured with Biebermax cream $t$ days from onset is given by the probability distribution $p(t)=m t+0.1$ for $0 \leq t \leq 20$ (where $m$ is a constant) and $p(t)=0$ for all other $t$.
(a) Find $m$. On average how soon do people clear up with treatment?
(b) If a hundred patients start applyng Biebermax, when will half of them clear up?

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | total (100) |
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