Final / 2014.12.15 / CS 3333.001 / Mathematical foundations of computer science

Name: _____

Please show all work and justify your answers.

1. Apply the extended Euclidean algorithm to find $\gcd(258, 192)$ and the Bézout coefficients.

2. Find all solutions x for each of the following systems of congruences. (a) $3x \equiv 4 \bmod 7$. (b) $x \equiv 2 \bmod 3$, $x \equiv 3 \bmod 5$, $x \equiv 2 \bmod 7$.

3. Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

4. Consider the linear system $3x + 4y = 7$, $x + y + z + 7 = 0$.

   (a) What augmented matrix A represents this system? Use Gauss-Jordan elimination to find its reduced row echelon form. Show steps.

   (b) Use (a) to find all solutions to the system in terms of the free variable(s). Check your answer. Sketch the solution set.

5. For each of the following parts give a concrete example of a 2×2 real non-diagonal matrix A satisfying the given conditions. (a) $A^2 = 0$. (b) A is upper triangular with two distinct eigenvalues. (c) A has exactly one (double) eigenvalue, which is not zero. (d) A does not commute with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

6. (a) Find the characteristic polynomial of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, its roots — the eigenvalues of A, and corresponding eigenvectors.

   (b) Find an invertible matrix P such that $P^{-1}AP$ is diagonal. Check your answer. Sketch the eigenspaces and describe the geometrical effects of the plane transformation $x \mapsto Ax$.

7. How many permutations of the 16 hex digits contain the strings **42**, **bad**, and **def**?

8. At a co-ed tennis doubles tournament each player is as likely to be a lady as a gent. When a random pair is picked, what is the probability of the event that the team is mixed? What is the probability of the event that the team has at most one gent? Determine, with proof, whether the above two events are independent.

9. Let t be the number of days it takes for one of Justin Bieber's tattoos to heal. Assume that $c > 0$ and the probability density function for t is $p(t) = c\,(1 - 0.025\,t)$ for $0 \le t \le 40$ and 0 otherwise.

   (a) Sketch $p(t)$ on the interval $-10 \le t \le 50$. What value of c makes $p(t)$ into a probability density? Use this value of c to answer the following questions.

   (b) What is the likelihood that a random tattoo will heal within 10 days?

   (c) How long does it take on average for Justin's tattoos to heal?

   (d) If Justin goes wild and gets lots of tattoos at once, how many days will it take for half of them to heal?

1	2	3	4	5	6	7	8	9	total (90)