ADVANCED EXAMINATION / TOPOLOGY / Fall 2011

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT SAN ANTONIO

Name: ______________________________

#	#	#	#	#	#	#	#	total (80)

Work any 8 problems. Indicate which problems you are doing in the top parts of the boxes above. Please show all work and either name or state the results you use in your proofs.

1. Show that the union of a family of pairwise non-disjoint (i.e. any two of them meet) connected subsets of a topological space is connected. Use this to prove that the product of two connected spaces is connected.

2. Prove that the co-finite topology on an infinite set X is T_1, but not T_2. With this topology is X connected? Compact? Prove your assertions.

3. Prove that a finite product of T_2 spaces is T_2.

4. Suppose $A_\alpha \subseteq X_\alpha$ are closed. Prove that $\prod_\alpha A_\alpha$ is closed in $\prod_\alpha X_\alpha$. Conclude that in general, the product of closures is the closure of the product.

5. Prove that a space is compact if, and only if, each filter has a cluster point. State (without proof) an analogous characterization of compactness using nets. Use it to prove Tychonoff's theorem about products of compact spaces.

6. Prove that a convex subset of Euclidean space is contractible. Prove that a contractible space is path connected. Prove that an open connected subset of Euclidean space is path connected.

7. Prove that the unit sphere in Euclidean space is homotopy equivalent to the punctured Euclidean space. Are these spaces homeomorphic? Prove your assertion for the special case S^1 and $\mathbf{C}^*$. What conclusion can you draw about the first simplicial homology group of $\mathbf{C}^*$?

8. Prove that the 0-th homology group of a path connected space is isomorphic to $\mathbf{Z}$. What happens if the space has several path connected components?

9. Let $z_0 = z_4 = 1, z_1 = i, z_2 = -1, z_3 = -i \in \mathbf{C}$, and let $\sigma_k(s) = (1-s)z_k + sz_{k+1}$. Combine these 1-simplices into a 1-chain $\sigma = \sum_{k=0}^{3} \sigma_k$. Prove that σ is a cycle. Find a 2-chain in $\mathbf{C}$ whose boundary is σ. Is it possible to find such a 2-chain in $\mathbf{C}^*$? What does that say about the first simplicial homology group of $\mathbf{C}^*$?

10. Cover the torus with a finite family of contractible open sets, whose mutual intersections are also contractible, and compute the first Čzech cohomology group for this cover.