ADVANCED EXAMINATION · COMPLEX ANALYSIS · Spring 2004

Name: ____________________

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Choose 8 questions to answer. Enter the selected problems in the top parts of the boxes above. Please supply brief narration with your formulas, state the results you use, and show all work!

1. Suppose f is holomorphic on a domain. Prove that f is constant
   (a) if the imaginary part $\Im[f]$ is constant.
   (b) if the modulus $|f|$ is constant.
2. Evaluate the following integrals around the unit circle: (a) $\displaystyle\int \frac{dz}{\tan z}$ (b) $\displaystyle\int \frac{dz}{z^3 + 3iz^2}$
3. Expand $f(z) = \dfrac{1}{z^2 + 5iz - 6}$ in a Laurent series valid in the annulus $\{z \in \mathbf{C}\colon 2 < |z| < 3\}$.
4. Construct a Möbius transformation taking the imaginary axis to the unit circle. Prove that the transformation you constructed does what is claimed. State the various properties of Möbius transformations that you use in your proof.
5. Suppose f is analytic in a punctured open disc D^* centered at the origin. What is the relationship between the integral of $f(z)\,dz$ around a circle in D^* centered at 0 (called the residue of f at 0) and the Laurent expansion of f at 0? Prove your assertion.
6. Suppose f is entire. Prove that $M(r) = \max\limits_{|z|=r} |f(z)|$ is a nondecreasing function of r.

   What can you conclude about f, if $M(r_1) = M(r_2)$ for some $r_1 \neq r_2$?
7. Suppose g is holomorphic in a neighborhood of the origin and $g(0) = g'(0) = 0$.

   Show that $h(z) = z^{-2}g(z)$ has a removable singularity at 0.
8. Prove the fundamental theorem of algebra: every nonconstant polynomial with complex coefficients has a complex root. You may use one of the following: (a) Liouville's theorem, (b) the maximum modulus principle, (c) Rouché's theorem.
9. State and prove the Riemann extension theorem on removable singularities.
10. State and prove the principle of analytic continuation.
11. State and prove the theorem of Weierstrass on convergent sequences of analytic functions.
12. Determine the number of zeros of $f(z) = z^5 + 3z - 2$ in the annulus $\{z\colon 1 < |z| < 2\}$.

    Justify your assertion.