ADVANCED EXAMINATION / ALGEBRA / Summer, 2014

THE UNIVERSITY OF TEXAS AT SAN ANTONIO

Name: ______________________________

#	#	#	#	#	#	#	#	total (80)

Work any 8 problems. Indicate which problems you are doing in the top parts of the boxes above. Please show all work and either name or state the results you use in your proofs.

Notation: K: commutative ring with unity, D: integral domain, F: field, m, n: positive integers.

1. Suppose $\operatorname{char} F \neq 2$ and M is a two-dimensional vector space over F. Given a basis for M, find bases for $M \otimes_F M$ and $\operatorname{Alt}_2(M)$.

2. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence of K-modules and C is free. Prove that $B \cong A \oplus C$.

3. If X is a set, find the K-module dual to the free K-module generated by X.

4. Prove that the direct sum of a nonempty family of free K-modules is free.

5. Suppose G is an additive group, $m > 1$, and $H = \{g \in G : mg = 0\}$. Prove that $H < G$, $\operatorname{Hom}(\mathbf{Z}, G) \cong G$, and $\operatorname{Hom}(\mathbf{Z}_m, G) \cong H$. If $G = \mathbf{Z}_n$ prove that H is cyclic. What is its size?

6. Verify the class equation for S_5 by computing the sizes of conjugacy classes.

7. Let $p(x) = x^2 + 5x + 1$, $F = \mathbf{Q}[x]/\langle p \rangle$ (why is F a field?), and $u = x + \langle p \rangle \in F$. Express u^4 as a linear combination of 1 and u and find its minimal polynomial. Same for u^{-1}.

8. Suppose $f\colon A \to B$ is a K-module morphism. State and prove the universal property of the canonical projection $p\colon B \to \operatorname{coker} f$.

9. Let $d = \gcd(m, n)$. Prove $[x, y] \mapsto xy$ defines a universal $\mathbf{Z}$-bilinear map $\mathbf{Z}_m \times \mathbf{Z}_n \to \mathbf{Z}_d$.

10. Prove that the set of units $U(K)$ is a multiplicative group. Prove that K is a local ring if and only if $K \setminus U(K)$ is an ideal.

11. What are the units of the ring of all functions $\mathbf{R} \to \mathbf{R}$ with pointwise addition and multiplication? What are its zero divisors? Nilpotent elements? Let $c \in \mathbf{R}$. Show that $\{f\colon \mathbf{R} \to \mathbf{R}\colon f(c) = 0\}$ is a maximal ideal. Exhibit a maximal ideal that is not of this form.